Honours:Selfish Routing and the Price of Anarchy/Chapter 2

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Contents

The model

- A source-destination pair $\{s_i, t_i\}$ is a *commodity*.
- A graph with one or more source-destination pairs is a *network*.
- $\mathcal{P}_i$ is the set of all paths for commodity *i*, and $\mathcal{P} = \bigcup_i \mathcal{P}_i$
- A *flow* as the aggregated routes chosen by a large number of agents; f_P measures the amount of agents that use the route *P*.
- The amount of flow using paths that include edge e is:

$$f_e = \sum_{P \in \mathcal{P} : e \in P} f_P$$

- A flow *f* induces a *flow on edges* $\{f_e\}_{e \in E}$
- A flow induces a unique flow on edges, but a flow on edges can admit many different path decompositions.
- The amount of traffic for a commodity *i* is $0 < r_i < \infty$.
- A flow *f* is *feasible* if for all *i*:

$$\sum_{P \in \mathcal{P}_i} f_P = r_i$$

- Edges have a cost, $c_e()$, which can depend on how much flow is on the edge.
- An *instance* is a triple of the form (G, r, c).
- The *cost of a path P with respect to a flow f* is the sum of the cost of the edges in the path:

$$c_P(f) = \sum_{e \in P} c_e(f_e)$$

- The *cost of a flow f* in a particular instance is the sum of all costs incurred by the traffic:

$$C(f) = \sum_{P \in \mathcal{P}} c_P(f) f_P = \sum_{e \in E} c_e(f_e) f_e$$

- A flow is *optimal* if it minimizes $C(f)$ over the set of feasible flows.

Flows at Nash equilibrium

- A flow is an equilibrium iff whenever traffic switches paths, the cost incurred by that traffic can only increase.

That is, a flow *f* feasible for the instance (G, r, c) is at *Nash equilibrium* or is a *Nash flow*, if for all commodities $i \in \{1, \ldots, k\}$, paths $P_1, P_2 \in \mathcal{P}_i$ with $f_{P_1} > 0$, and all amounts $\delta \in (0, f_{P_1}]$ of traffic on P_1:

$$c_{P_1}(f) \leq c_{P_2}(\tilde{f})$$

where the flow $\tilde{f}$ is obtained from *f* by moving δ units of flow from the path P_1 to P_2.

- In a flow at Nash equilibrium, all flow travels on *minimum-cost paths*. That is, a flow *f* is at Nash equilibrium iff $c_{P_1}(f) \leq c_{P_2}(f)$ (same conditions as above).
- If *f* is a flow at Nash equilibrium for an instance, for each commodity *i*, all paths of *f* have a common cost $c_i(f)$. Hence:

$$C(f) = \sum_{i=1}^{k} c_i(f) r_i$$

- *Existence:* Every instance admits a Nash flow.
- *Uniqueness:* All Nash flows of an instance have the same cost. That is, if f and $\tilde{f}$ are flows at Nash equilibrium for an instance, then $C(f) = C(\tilde{f})$.

The price of anarchy

- The *price of anarchy* is the worst-possible ratio between the cost of a flow at Nash equilibrium and that of an optimal flow:

$$\rho(G, r, c) = \frac{C(f)}{C(f^*)}$$

- An optimal flow can be found via a non-linear program:
  - Min $\sum_{e \in E} h_e(f_e)$ where $h_e(f_e) = c_e(f_e) f_e$
  - Subject to:
    - $\sum_{P \in \mathcal{P}_i} f_P = r_i \forall i \in \{1, \ldots, k\}$
    - $f_e = \sum_{P \in \mathcal{P}: e \in P} f_P \forall e \in E$
    - $f_P \geq 0 \forall P \in \mathcal{P}$
- A *convex combination* of *x* and *y* is a point on the line segment between *x* and *y*: $\lambda x + (1 - \lambda) y$ for some $\lambda \in [0, 1]$.
- A subset *S* is *convex* if it contains all of its convex combinations.
- A function, defined on a convex subset, is *convex* if all line segments between two points on the graph lie above the graph.
- A function $c : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is *semiconvex* if the function $x \cdot c(x)$ is convex.
- Optimal and Nash flows are the same thing, just with different cost functions.
- If *c* is a differentiable cost function, the *marginal cost function* is:

$$c^* = \frac{d}{dx}(x \cdot c(x)) = c(x) + x \cdot c'(x)$$

and this can be interpreted as taking into account the cost experienced by the new traffic, and the cost incurred by the existing traffic due to the additional congestion.

- A flow *f* feasible for (G, r, c) is optimal iff it is at Nash equilibrium for the instance (G, r, c^*).
- If an instance has semiconvex cost functions, then an optimal flow for it can be computed, up to an arbitrarily small error term, in time polynomial in the size of the instance and the number of bits of precision required.
- If $P \neq NP$ there is no algorithm for computing an approximation of an optimal flow in arbitrary instances (removing the semiconvexity assumption) that always runs in polynomial time.

Existence and uniqueness of Nash flows

- A Nash flow exists by a solution of linear program, and all Nash flows have the same cost.
- The property of being at Nash equilibrium is a property only of the induced flow on edges, and not of the particular path decomposition.
- If you have a strictly increasing cost function c_e, if f and $\tilde{f}$ are Nash flows, then $f_e = \tilde{f}$.
- A feasible flow *f* is at Nash equilibrium iff:

$$\sum_{e \in E} c_e(f_e) f_e \leq \sum_{e \in E} (f_e) f_e^*$$

- A Nash flow for an instance can be computed, up to an arbitrarily small error term, in time polynomial in the size of the instance and the number of bits of precision required.

Nash flows in single-commodity networks

- Every single-commodity instance admits a directed acyclic Nash flow.
- Let *f* be a flow feasible for an instance. For a vertex *v* in *G*, let *d(v)* denote the length, with respect to edge lengths $c_e(f_e)$, of a shortest *s*-*v* path in *G*. Then:

  $$d(w) - d(v) \leq c_e(f_e)$$

  for all edges *e* = *(v, w)*, and *f* is at Nash equilibrium iff equality holds whenever $f_e > 0$
- If *f* is a directed acyclic Nash flow for a single-commodity instance, then there is an *f*-monotone ordering of the vertices in *G*.

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