Honours:Selfish Routing and the Price of Anarchy/Chapter 3

From Nointrigue

< Honours:Selfish Routing and the Price of Anarchy

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The price of anarchy with linear cost functions

Preliminaries

- Each edge e is linear in the edge congestion: $c_e(x) = a_e(x) + b_e$ for some $a_e, b_e \geq 0$.
- The total cost $C(f)$ of a flow *f* is:

$$C(f) = \sum_{e \in E} a_e f_e^2 + b_e f_e$$

- This is a convex quadratic function and the NLP is a convex program.
- With a linear cost function:
  - A feasible flow *f* is at Nash equilibrium iff for each commodity *i* and paths $P_1, P_2 \in \mathcal{P}_i$ with $f_{P_1} > 0$:

$$\sum_{e \in P_1} a_e f_e + b_e \leq \sum_{e \in P_2} a_e f_e + b_e$$

  - A feasible flow f^* is optimal iff for each commodity *i* and paths $P_1, P_2 \in \mathcal{P}_i$ with $f^*_{P_1} > 0$:

$$\sum_{e \in P_1} 2a_e f_e^* + b_e \leq \sum_{e \in P_2} 2a_e f_e^* + b_e$$

- If cost functions are of the form $c_e(x) = a_e x$, then optimal iff Nash equilibrium

- Suppose instance has linear cost functions, and *f* is a Nash flow:
  - For each edge *e*, $c_e^*(f_e/2) = c_e(f_e)$
  - The flow *f*/2 is optimal for $(G, r/2, c)$

Proof of upper bound

- If an instance has linear cost functions, then $\rho(G,r,c) \le \frac{4}{3}$.
- This was proved using two lemmas:
  - $C(f/2) \ge \frac{1}{4} \cdot C(f)$
  - For every $\delta > 0$, a feasible flow for the instance $(G,(1+\delta)r,c)$ has cost at least $C(f^*) + \delta \sum_{e \in E} c_e^*(f_e^*) f_e^*$
- Pigou's example shows that the upper bound of 4/3 cannot be improved; the worst-case inefficiency due to selfish routing can always be explained with the simplest of networks.

A general upper bound on the price of anarchy

The anarchy value

- If *c* is a cost function, the *anarchy value* $\alpha(c)$ of *c* is:

$$\alpha(c) = \sup_{x,r \ge 0} \frac{r \cdot c(r)}{x \cdot c(x) + (r-x)c(r)}$$

- The *anarchy value* $\alpha(\mathcal{C})$ of a set $\mathcal{C}$ of cost functions is:

$$\alpha(\mathcal{C}) = \sup_{0 \ne c \in \mathcal{C}} \alpha(c)$$

- The anarchy value captures how ill-behaved a set *C* of allowable cost functions is.
- The anarchy value of a set lies in $[1,\infty]$.
- If *C* contains the constant functions, then $\alpha(\mathcal{C})$ *lower bounds* the price of anarchy for instances with cost functions in $\mathcal{C}$.

Proof of the upper bound

- Let *C* be a set of cost functions with anarchy value $\alpha(\mathcal{C})$, and (G,r,c) an instance with cost functions in *C*. Then:

$$\rho(G,r,c) \le \alpha(\mathcal{C})$$

Matching lower bounds in simple networks

- For a set *C* of cost functions that contains the constant functions, the worst possible value of $\rho(G,r,c)$ for a multicommodity instance with cost functions in *C* is met by a single-commodity instance on a two-node, two-link network (up to an arbitrarily small additive factor).
- A set *C* of cost functions is *diverse* if for each positive scalar $\gamma > 0$, there is a cost function $c \in \mathcal{C}$ such that $c(0) = \gamma$. For a set of cost functions that is closed under multiplication by positive scalars, diversity merely asserts that some cost function is positive when evaluated at 0.
- If we relax the assumption of constant functions, and assume diversity, the worst-case ρ value is achieved by a single-commodity instance on a network of parallel links.
- Worst-case examples are simple for broader sets of cost functions.
- No nontrivial restriction on the allowable network topologies reduces the price of anarchy.

Computing the price of anarchy

The price of anarchy with polynomial cost functions

- If $\mathcal{I}_p$ is the set of instances with cost functions that are polynomials up to a degree of *p*, then:

$$\sup_{(G,r,c)\in\mathcal{I}_p} \rho(G,r,c) = \Theta\left(\frac{p}{\ln p}\right)$$

- That is, the price of anarchy is small unless cost functions are extremely steep.

The price of anarchy with M/M/1 delay functions

- Cost functions of the form $c(x) = (u-1)^{-1}$ for $x < u$ arise as the delay function of an M/M/1 queue.
- This function models an edge of capacity *u*, with the delay of the link approaching infinity as the amount of traffic approaches the capacity.
- Let R_{max} be the largest allowable sum of all traffic rates, and u_{min} be the smallest allowable edge capacity.
- $\alpha(c) = \frac{1}{2}\left(1 + \sqrt{\frac{u}{u - R_{max}}}\right)$
- Selfish routing is benign when there is excess capacity.

A bicriteria bound in general networks

- Generally, if no restrictions are placed on the cost functions of a network, then the price of anarchy is unbounded, even in simple networks.
- Instead of directly comparing the costs of Nash and optimal flows, we compare the cost of a flow at Nash equilibrium to that of an optimal flow that must route additional traffic.
- If *f* is a flow at Nash equilibrium for (G, r, c), and f^* is feasible for $(G, 2r, c)$, then $C(f) \leq C(f^*)$.
- If *f* is a flow at Nash equilibrium for (G, r, c) and f^* is feasible for $(G, (1+\xi)r, c)$ with $\xi > 0$, then $C(f) \leq \frac{1}{\xi} \cdot C(f^*)$.
- For networks with M/M/1 delay functions, to beat optimal routing, double the capacity of every edge.
