Honours:Selfish Routing and the Price of Anarchy/Chapter 4

From Nointrigue

< Honours:Selfish Routing and the Price of Anarchy

Contents

Nonatomic congestion games

- This is a generalisation of the preceding traffic models onto games that don't involve networks.
- Basic definitions:
  - Finite ground set *E* of *elements*
  - Each element has a *cost function* c_e
  - *k* player types
  - The *strategy set* of players of type *i* is a finite multiset $\mathcal{S}_i$ of subsets of *E*
  - The continuum of players of type *i* is represented by an interval $[0, n_i]$
  - The amount of congestion contributed to element *e* by players of type *i* selecting strategy *S* is the *rate of consumption* $a_{S,e} > 0$
- A *nonatomic congestion game* is defined by a 5-tuple $(E, c, \mathcal{S}, n, a)$
- Mapping back to the traffic routing model:
  - Resources = network edges
  - Player types = commodities
  - Strategy sets = collections of source-destination paths
  - Rate of consumption = 1
- A (feasible) *action distribution* is a vector *x* of nonnegative real numbers with components indexed by the disjoint union of $\mathcal{S}_1, \ldots, \mathcal{S}_2$ (i.e. one element in the vector for each strategy for each of the players) with the property that $\sum_{S \in \mathcal{S}_i} x_S = n_i$ for each player type *i*.
- For the total amount of congestion induced on element *e* by the action distribution *x*:

$$x_e = \sum_{i=1}^{k} \sum_{S \in \mathcal{S}_i : e \in S} a_{S,e} x_S \,.$$

- Players of type *i* selecting a strategy *S* incur a cost:

$$c_S(x) = \sum_{e \in S} a_{S,e} c_e(x_e)$$

- Cost of an action distribution *x* is the total disutility experienced by all the players:

$$C(x) = \sum_{i=1}^{k} \sum_{S \in \mathcal{S}_i} c_S(x) x_S$$

- In general, the results for the price of anarchy of the traffic model carry through to NCGs because the combinatorial structure of the underlying network and the uniformity of rates of consumption were never used.

Approximate Nash flows

- A flow *f* feasible for an instance is at ϵ *-approximate Nash equilibrium* if for all commodities *i*, $P_1, P_2 \in \mathcal{P}_i$ with $f_{P_1} > 0$, and $\delta \in (0, f_{P_1}]$:

$$c_{P_1}(f) \leq (1+\epsilon) c_{P_2}(\tilde{f})$$

- That is, the path chosen by an agent does not need to have minimum cost, but its cost cannot exceed the minimum by more than a factor of $(1+\epsilon)$.
- You can extend the previous results to cater for approximate flows.
- Upper bound of price of anarchy for networks with linear cost functions: If $\epsilon \in [0, 3)$, *f* is at ϵ-approximate Nash equilibrium for an instance with linear cost functions, and f^* is feasible for the instance, then:

$$C(f) \leq \frac{4+4\epsilon}{3-\epsilon} \cdot C(f^*)$$

- This bound cannot be improved, and this can be shown by using an example as before.

Edge capacities

Atomic selfish routing

- We now have a finite number of agents, each of which controls a nonnegligible amount of flow.

Splittable flow

- There are *k* users, where each user *i* intends to send r_i units of flow from the source to the destination.
- A flow *f* consists of *k* nonnegative real vectors, one for each user.
- The feasible flows in an atomic splittable instance are isomorphic to those in the corresponding nonatomic instance.
- The total cost experienced by user *i* is:

$$C_i(f) = \sum_{P \in \mathcal{P}_i} c_P(f) f_P^{(i)}$$

- A flow *f* feasible for an atomic splittable instance is at Nash equilibrium if for each agent *i*, $f^{(i)}$ minimises $C_i(f)$ while holding $f^{(j)}$ fixed for every $j \neq i$.
- Every atomic splittable instance admits a flow at Nash equilibrium.
- The difference between this case and the nonatomic case is that in the atomic case, the agent will not take into account the congestion it causes for its own traffic, and therefore agents partially internalise the social loss caused by their traffic.
- The bicriteria bound can be extended to atomic splittable instances.
- The upper bound on the price of anarchy carries over to the atomic splittable case as well.

Unsplittable flow

- Each agent must route all its flow on one path.
- The cost of a Nash flow in an atomic unsplittable case can be arbitrarily larger than that of flow routing twice as much traffic.
- The upper bounds on the price of anarchy fail to carry over to this case.

- Nash flows are not unique in atomic unsplittable instances.
- The *best* Nash flow obeys both the bicriteria bound and the upper bound on the price of anarchy (at least if all cost functions are semiconvex and all commodities share the same source vertex).
- In atomic unsplittable instances with cost functions that are polynomials with degree at most *d* and non-negative coefficients, the price of anarchy is bounded above by a constant that is exponential in *d* but independent of the network size and the number of commodities.

A quick-and-dirty bound on the price of anarchy

Better bounds for many traffic rates

Maximum cost
