Honours:Algorithmic Aspects of Game Theory

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Lecture 2

Sperner's Lemma

Take a triangle and number the vertices 0, 1 and 2. Triangulate it into smaller triangles, and each of the vertices of the triangulation is numbered as follows:

- A vertex on a side of the simplex cannot be assigned the same number as the corner of the simplex opposite that side
- Vertices inside the simplex can be labelled with any of the numbers.

Lemma: You will always have a small triangle somewhere in the simplex with its vertices labelled (0, 1, 2). Moreover, you will have an odd number of such triangles.

Brouwer's Fixed Point Theorem

Using Sperner's Lemma, we can prove Brouwer's Fixed Point Theorem.

Theorem: Let S be any n-dimensional simplex and let $\phi : S \longrightarrow S$ be any continuous function. Then ϕ has a fixed point, i.e. $\exists x^* \in S$ such that $\phi(x^*) = x^*$.

Arrow-Debreu Theorem (General Equilibria Exist)

Suppose we have n agents and k different commodities. Each agent i has an endowment $e_i \in \mathbb{R}_+^k$, and a utility function $u_i : \mathbb{R}_+^k \longrightarrow \mathbb{R}_+$. Agents go to a market and exchange goods with other agents to maximise their overall utility. The Pareto Point is where no agent can improve their overall utility through further exchanges. When the market clears, this means that all the goods for sale have been purchased, that is, $\hat{X}(p) \leq E$, where $\hat{X}(p)$ describes the total amounts of each commodity demanded by all agents, and E describes the total amount of goods brought to the market.

Theorem: There is always a price p^* such that $\hat{X}(p) \leq E$, i.e. one can always find a price that clears the market. Such a price is a general equilibrium. This price is also a *Pareto Point*.

Kakutani's Fixed Point Theorem

This is a generalisation of Brouwer's Fixed Point Theorem.

Theorem: Let $\Phi : S \longrightarrow 2^S$ be any convex valued function such that for any sequence (x_i, y_i) converging to (x, y), if $y_i \in \Phi(x_i) \forall i \in \mathbb{N}$, then $y \in \Phi(x)$ (this property is called graph continuity). Then Φ has a fixed point, i.e. $\exists x^* \in S$ such that $x^* \in \Phi(x^*)$.

Nash Equilibria Exist

Mixed strategies occur when a player chooses a move $s \in S$ randomly according to some distribution $\prod$ on S. If player i chooses from a distribution $\prod_i$ on S_i, each player tries to maximise:

$$p_i(\Pi_1, \Pi_2) = \sum_{s_1 \in S_1, s_2 \in S_2} \Pi_1(s_1)\Pi_2(s_2)p_i(s_1, s_2)$$

A tuple of mixed strategies is said to be in mixed Nash equilibrium if:

$$\forall \Pi_1' \text{on} S_1 : p_1(\Pi_1, \Pi_2) \geq p_1)(\Pi_1', \Pi_2)$$
$$\forall \Pi_2' \text{on} S_2 : p_2(\Pi_1, \Pi_2) \geq p_1)(\Pi_1, \Pi_2')$$

Every game has a mixed strategy Nash equilibrium. (For many games, however, it is the case that no pure Nash equilibrium exists.)

Proof

Let $\Phi_1(\Pi_2)$ be the set of all mixed strategies $\Pi_1' \in D(S_1)$ that maximise $p_1(\Pi_1', \Pi_2)$ for a mixed strategy $\Pi_2 \in D(S_2)$. Similarly, define the set $\Phi_2(\Pi_1)$. Let $\Phi(\Pi_1, \Pi_2) = \Phi_1(\Pi_2) \times \Phi_2(\Pi_1)$.

$\Phi : D(S_1) \times D(S_2) \longrightarrow 2^{D(S_1) \times D(S_2)}$ is convex valued and graph continuous. Hence, by Kakutani's fixed point theorem, Φ has a fixed point (Π_1^*, Π_2^*), which is a mixed Nash equilibrium.

Lecture 3

Extensive games

- An extensive game is a game where the players take turns.
- An extensive game can be represented by a decision tree, and the game starts at the root and finishes at one of the leaves.
- An extensive game be translated to strategic (standard) form.

Repeated games

- A repeated game is a standard game which is played repeatedly.
- Backwards induction can be used to derive the optimum strategy if the number of rounds is finite, but when the players have only bounded rationality, different equilibria may emerge.

Games with incomplete information

- Players have to take actions without having full information on the different factors that influence their utility.
- Ω is the *set* of possible states of the world. p and q are the player's beliefs as to the state of the world (these are probability distributions on Ω.
- Each player gets some signal as to the real state of the world. S and T are the sets of possible signals. Each state of the world gets turned into a signal: $f : \Omega \rightarrow S$ and $g : \Omega \rightarrow T$.
- X and Y are the strategy spaces for each player.
- When p = q and $U_i(x, y, \omega) = U_i(x, y)$, the game is called a *correlation game*. The state of the world does not affect the utilities of the players. Through signals, they can correlate actions.

Zero sum games

- A zero sum game is a special case of a standard game where $U_1(x, y) + U_2(x, y) = 0$ for any pair of strategies (x, y). Let $\bar{X}$ and $\bar{Y}$ be the sets of all possible mixed strategies of players 1 and 2 respectively.
- x^* is a max-minimiser for player 1 iff:

$$\min_{y \in \bar{Y}}(U_1(x^*, y)) \geq \min_{y \in \bar{Y}}(U_1(x, y)) \, \forall x \in \bar{X}$$

- Player 1 should choose a max-minimiser for himself if player 2 knows which mixed strategy player 1 chose before choosing his own strategy.
- The max-minimiser for player 1 can be formulated as a LP: max Z s.t. $\sum_x pi_x U_1(x, y) \geq Z \, \forall y$, and $\sum_x \pi_x = 1$
- Hence, a Nash equilibrium can be found efficiently in this class of games. The max-minimisers for players 1 and 2 are equal due to the duality theorem.
- In general, there is no efficient algorithm for finding a Nash equilibrium.
- For games restricted to two players, all Nash equilibria are always rational numbers.

Dominance and rationality

- A pure strategy $x > x'$ if $u_1(x, y) > u_1(x', y), \forall y \in \bar{Y}$.
- In Prisoner's Dilemma, D dominates C.
- *Iterated dominance:* list all pure strategies for both players, and while either player has a dominated strategy, eliminate it.
- **Theorem:** The result does not depend on the order of elimination.
- A strategy is *rationalizable* if it is the best response to some mixed strategy of a subset of the other player's strategies.

- **Theorem:** In two player games, a strategy is rationalizable iff it survives iterated dominance.
- x *weakly dominates* x' if $\forall y u_1(x,y) \geq u_1(x',y)$ and $u_1(x,y) > u_1(x',y)$ for some y.
- You can create finite state automata to represent how the players should choose the strategies.

Subgame perfect equilibria

- An equilibrium is *subgame perfect* if when played from any point in the game, it is a Nash equilibrium. A *subgame* is a subset of nodes that still form a game.

Lecture 4

Social choice theory

- Given a set of individual preferences, how do we make the best choice globally?
- Suppose there are n agents and a set of social choices $C = \{c_1, c_2, \ldots, c_m\}$ The preference relation $\geq_i$ over C is defined as the ordering of set C according to the preference of agent i.
- A social welfare functional is a function $\geq (\geq_1, \ldots, \geq_n)$ that translates individual preferences into a global preference.
- *Condercet paradox:* majority voting on pairs can lead to cyclical preferences
- **Theorem: (Arrow's Impossibility Theorem)** Let $|C| \geq 3$. Then there is no social welfare functional $>$ that satisfies all of the following properties:
  1. $>$ is independent from irrelevant alternatives
  2. $>$ respects unanimity
  3. $>$ is non-dictatorial
- **Alternative form of theorem:** Let $F(>_1, >_2, \ldots, >_n) \in C$ be a social choice function that simply chooses a winner based on the agents' preferences. Then, if F is monotonic, respects unanimity, and non-binary, then F must also be dictatorial.

Mechanism design

- The problem involves designing a game where selfish agents will ultimately play according to the desire of the game designer.
- We are given an *environment* (n, C, U), where:
  - n is the number of agents/players
  - C denotes the set of possible outcomes
  - U is a set of tuples of the form $(u_1, \ldots, u_n)$, $u_i : C \rightarrow \mathbb{R}$, i.e. the set of all possible combinations of agent utilities.
- Choice rule: $f : U \rightarrow 2^C$, which maps a combination of agent utilities to a set of outcomes.
- The goal of mechanism design is to design a game with sets of strategies $X_1, \ldots, X_n$ and an outcome function $g : X_1 \times \cdots \times X_n \rightarrow C$, such that the utility of player i in the game is $u_i(x_1, \ldots, x_n) = u_i(g(x_i, \ldots, x_n))$ and the resulting game G satisfies the following two conditions:
  - For each player there exists a weakly dominating strategy
  - If $(x_1, \ldots, x_n) \in \mathrm{Dom}(G)$, then $g(x_1, \ldots, x_n) \in f(u_1, \ldots, u_n)$