Integration (Mathematics Extension 2)

This document contains notes about Extension 2 Integration instead of providing a summary. Examples are given.

Reduction to a standard form

When the integrand is a rational function of the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials, use the division transformation and partial fractions.

Using implicit differentiation

Using implicit differentiation on something like $u^2 = x^2 + 1$ is easier than deriving $u = \sqrt{x^2 + 1}$ to find $\frac{du}{dx}$. When substitutions like $u^2 = x^2 + 1$ are made, it is necessary to restrict the values of u so that each x-value in the domain of the integrand corresponds to exactly one value of u.

$$u^2 = x^2 + 1$$
$$2u\frac{du}{dx} = 2x$$
$$\therefore \frac{du}{dx} = \frac{x}{u}$$

Example

$$\int \frac{x^3}{\sqrt{x^2+1}}dx$$

$$u^2 = x^2 + 1 \qquad x^2 = u^2 - 1$$
$$2u\frac{du}{dx} = 2x \text{ (by implicit differentiation)}$$
$$\therefore dx = \frac{u}{x}du = \frac{u}{\sqrt{u^2-1}}du$$

$$\therefore \int \frac{x^3}{\sqrt{x^2+1}}dx$$
$$= \int \frac{(u^2-1)\sqrt{u^2-1}}{u} \cdot \frac{u}{\sqrt{u^2-1}}du$$
$$= \int (u^2-1)du$$
$$= \frac{u^3}{3} - u + c$$
$$= \frac{1}{3}u(u^2-3) + c$$
$$= \frac{1}{3}\sqrt{x^2+1}\left((x^2+1)-3\right) + c$$
$$= \frac{1}{3}\sqrt{x^2+1}(x^2-2) + c$$

Trigonometric substitutions

For integrands involving $\sqrt{a^2-x^2}$, use $x=a\sin\theta$ or $x=a\cos\theta$ to simplify the expression.
For $x=a\sin\theta$

$$\sqrt{a^2-x^2}=\sqrt{a^2\left(1-\sin^2\theta\right)}=\left|a\cos\theta\right|$$

Now, restricting $a>0$ and $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, each x in the domain of $\sqrt{a^2-x^2}$ corresponds to exactly one value of θ, and hence $\sqrt{a^2-x^2}=a\cos\theta$.

Substitution	Restriction	Result
$x=a\sin\theta$ or $x=a\cos\theta$	$a>0$ and $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$	$\sqrt{a^2-x^2}=a\cos\theta$
$x=a\tan\theta$	$a>0$ and $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$	$\sqrt{a^2+x^2}=a\sec\theta$
$x=a\sec\theta$	$a>0$ and $-\pi<\theta\le-\frac{\pi}{2}$ or $0\le\theta<\frac{\pi}{2}$	$\sqrt{x^2-a^2}=a\tan\theta$

Example

$\int\frac{dx}{x^2\sqrt{x^2-4}}$ using substitution $x=2\sec\theta$

$$x=2\sec\theta$$

$$dx=2\sec\theta\tan\theta d\theta$$

$$\begin{aligned}
&\therefore\int\frac{dx}{x^2\sqrt{x^2-4}}\\
&=\int\frac{2\sec\theta\tan\theta d\theta}{4\sec^2\theta\sqrt{4\sec^2\theta-4}}\\
&=\int\frac{2\sec\theta\tan\theta}{4\sec^2\theta\cdot2\tan\theta}d\theta\\
&=\frac{1}{4}\int\frac{1}{\sec\theta}d\theta\\
&=\frac{1}{4}\int\cos\theta d\theta\\
&=\frac{1}{4}\sin\theta+c
\end{aligned}$$

$$\sin^2\theta=1-\cos^2\theta$$

$$\sec\theta=\frac{x}{2}$$

$$\therefore\cos\theta=\frac{2}{x}$$

$$\Rightarrow\sin^2\theta=\frac{x^2-4}{x^2}$$

$$\therefore\sin\theta=\frac{\sqrt{x^2-4}}{x}$$

$$\therefore\int\frac{dx}{x^2\sqrt{x^2-4}}=\frac{\sqrt{x^2-4}}{4x}+c$$

Integrating *sec θ and cosec θ*

$$\int \sec\theta d\theta$$
$$= \int \frac{\sec\theta(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta} d\theta$$
$$= \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta} d\theta$$
$$= \ln|\sec\theta + \tan\theta| + c$$

$$\int \operatorname{cosec}\theta d\theta$$
$$= \int \frac{\operatorname{cosec}\theta(\operatorname{cosec}\theta - \cot\theta)}{\operatorname{cosec}\theta - \cot\theta} d\theta$$
$$= \int \frac{-\operatorname{cosec}\theta\cot\theta + \operatorname{cosec}^2\theta}{\operatorname{cosec}\theta - \cot\theta} d\theta$$
$$= \ln|\operatorname{cosec}\theta - \cot\theta| + c$$

Using *t*-results

The substitution $t = \tan\frac{\theta}{2}$ can be used to convert an integrand involving trigonometric functions into a rational function of *t*.

$$\sin\theta = \frac{2t}{1+t^2} \qquad \cos\theta = \frac{1-t^2}{1+t^2} \qquad \tan\theta = \frac{2t}{1-t^2}$$

Integrals with Trigonometric Equations ($y = \cos^m x.\sin^n x$)

(Note: *m* or *n* is odd)

Example

$$\int \sin^5\theta\cos^4\theta d\theta$$
$$= -\int \sin^4\theta\cos^4\theta d(\cos\theta)$$
$$= -\int (1-\cos^2\theta)^2\cos^4\theta d(\cos\theta)$$
$$= -\int (\cos^4\theta - 2\cos^6\theta + \cos^8\theta) d(\cos\theta)$$
$$= -\left[\frac{1}{5}\cos^5\theta - \frac{2}{7}\cos^7\theta + \frac{1}{9}\cos^9\theta\right] + c$$
$$= -\frac{1}{5}\cos^5\theta + \frac{2}{7}\cos^7\theta - \frac{1}{9}\cos^9\theta + c$$

Using Sums and Products

$$2\sin a\cos b = \sin(a-b) + \sin(a+b)$$
$$2\cos a\cos b = \cos(a-b) + \cos(a+b)$$
$$2\sin a\sin b = \cos(a-b) - \cos(a+b)$$

Example

$$\int \sin 6x \sin 2x dx$$
$$= \frac{1}{2}\int (\cos 4x - \cos 8x) dx$$
$$= \frac{1}{2}\left[\frac{1}{4}\sin 4x - \frac{1}{8}\sin 8x\right] + c$$
$$= \frac{1}{8}\sin 4x - \frac{1}{16}\sin 8x + c$$

Integration by Parts

$$\int f(x)g(x)dx = F(x)g(x) - \int F(x)g'(x)dx$$

Examples

$$\int \sin^{-1} x dx$$
$$= x\sin^{-1} x - \int x \cdot \frac{1}{\sqrt{1-x^2}}dx$$
$$= x\sin^{-1} x - \int \frac{2x}{2\sqrt{1-x^2}}dx$$
$$= x\sin^{-1} x - \sqrt{1-x^2} + c$$

$$\int \ln x dx$$
$$= x\ln x - \int x \cdot \frac{1}{x} dx$$
$$= x\ln x - x + c$$

Using the Area of a Circle

Example

$\int_0^1 \sqrt{1-u^2}\,du = \frac{\pi}{4}$ (This is the area of $\frac{1}{4}$ of a circle with radius 1 unit)

Further Properties of Definite Integrals

1. For integrals such as $\int_{-2}^{2} \frac{x^2}{e^x + 1}dx$, use a substitution like $u = -x$.

2. $\int_{-a}^{a} f(x)dx = \int_0^a \{f(x) + f(-x)\}dx$

Proof

Using the substitution $u = -x$,

$$\int_{-a}^{0} f(x)dx$$
$$= -\int_a^0 f(-u)du$$
$$= \int_0^a f(-u)du$$
$$= \int_0^a f(-x)dx$$

$$\therefore \int_{-a}^{a} f(x)dx$$
$$= \int_0^a f(x)dx + \int_{-a}^{0} f(x)dx$$
$$= \int_0^a \{f(x) + f(-x)\}dx$$

3. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Proof

Let $u = a - x$

$du = -dx$

$x = 0 \Rightarrow u = a$

$x = a \Rightarrow u = 0$

$$\int_0^a f(x)dx$$
$$= -\int_a^0 f(a-u)du$$
$$= \int_0^a f(a-u)du$$
$$= \int_0^a f(a-x)dx$$