

Polynomials and Approximation of Roots (Extension 1)

Polynomials

- A *polynomial* is an expression that is the sum of coefficients multiplied by variables raised to non-negative integral powers. For example, $4x^3 + 8x^2 + x + 10$, x^{10} and 20 are polynomials, but $\frac{1}{5x^2+3}$, $x + x^{-1}$ and e^{2x} are not.
- In expanded form, a polynomial is given by:

$$\sum_{k=0}^n a_k x^k = a_n x^n + a_{n-1} x^{n-1} + \dots + a_k x^k + \dots + a_1 x + a_0 \quad (1)$$

where $a_n \neq 0$.

- A *term* of the polynomial is one part of a polynomial that consists of a number, which is called the *coefficient*, multiplied by the *variable* raised to a non-negative integral *power*. For example, the first term in the first example above is $4x^3$; the coefficient is 4, the variable is x and the power is 3.
- The *leading coefficient* is a_n and the *constant coefficient* is a_0 . The *leading term* is $a_n x^n$ and the *constant term* is a_0 . In the first example above, the leading term is $4x^3$ and the constant term is 10.
- The *degree* of the polynomial is largest power of any term, and it is written as $\deg(P)$. The degree of the first example above is 3.
- A *monic polynomial* is a polynomial with a leading coefficient of 1.
- Polynomials are infinitely differentiable¹, and their domain is \mathbb{R} .
- A sum (and difference) of polynomials is a polynomial.
- A product of polynomials is a polynomial.
- A quotient of polynomials is not necessarily a polynomial, however.
- The derivative and integral of a polynomial is a polynomial.

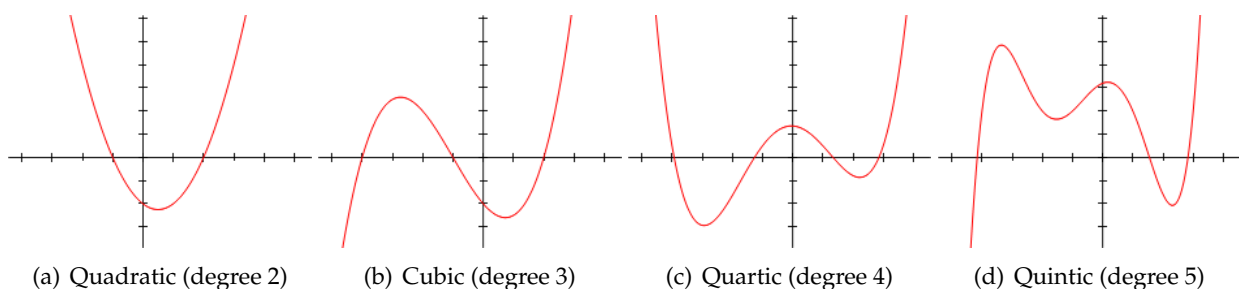


Figure 1: Some polynomials of low degree

¹That is, polynomials are analytic functions with an infinite radius of convergence.

Roots

- A *root* or *zero* of a polynomial is a number a such that $P(a) = 0$. a is also a *solution* of $P(x) = 0$.
- By the Fundamental Theorem of Algebra, a polynomial of degree n has n complex roots (which may be repeated). Hence, a polynomial of degree n has at most n unique, real roots.
- If x_1, x_2, \dots, x_n are the n roots of $P(x)$, the polynomial is given by:

$$P(x) = (x - x_1)(x - x_2) \dots (x - x_n) \quad (2)$$

- The *multiplicity* of a root is the number of times that the root is repeated. Formally, a is a root of multiplicity k if there exists a polynomial $S(x)$ such that $P(x) = (x - a)^k S(x)$ and a is not a root of $S(x)$. A root of multiplicity 1 is called a *simple root*.
- A polynomial with real coefficients will have:
 - An even (possibly zero) number of roots if the degree is even; or
 - An odd number of roots if the degree is odd.

This is a consequence of the fact that complex (non-real) roots come in conjugate pairs.

Approximation of Roots

Halving the Interval

In this case, $f(x)$ can be a polynomial or some other continuous function.

1. If $f(x_1)$ and $f(x_2)$ change signs, then clearly², the function must cross the x -axis and there must be at least one root in the interval between x_1 and x_2 . The first approximation to the root will be the average of x_1 and x_2 :

$$x_3 = \frac{x_1 + x_2}{2} \quad (3)$$

2. If $f(x_3) = 0$, then x_3 is a root.
3. If $f(x_3) \neq 0$, to get a better approximation, repeat the process using:
 - x_1 and x_3 if $f(x_1)$ and $f(x_3)$ have different signs; or
 - x_2 and x_3 if $f(x_2)$ and $f(x_3)$ have different signs.

Newton's Method

In this case, $f(x)$ can be a polynomial or some other differentiable function.

1. If x_1 is an approximation to the root, the next (hopefully better) approximation can be obtained by:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \quad (4)$$

2. Repeat, setting x_1 to be x_2 .

Newton's method can also be used to approximate a local minimum or maximum of a function by applying it to the first derivative, because stationary values are given by the solutions to $f'(x) = 0$.

²... or by an application of the Intermediate Value Theorem.

Relationships between the Roots

Quadratics

If α and β are the roots of $P(x) = ax^2 + bx + c$, then:

$$\text{Sum of roots} = \alpha + \beta = -\frac{b}{a} \quad (5)$$

$$\text{Product of roots} = \alpha\beta = \frac{c}{a} \quad (6)$$

Cubics

If α, β and γ are the roots of $P(x) = ax^3 + bx^2 + cx + d$, then:

$$\text{Sum of roots} = \sum \alpha = \alpha + \beta + \gamma = -\frac{b}{a} \quad (7)$$

$$\text{Sum of roots taken two at a time} = \sum \alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} \quad (8)$$

$$\text{Product of roots} = \prod \alpha = \alpha\beta\gamma = -\frac{d}{a} \quad (9)$$

Quartics

If α, β, γ and δ are the roots of $P(x) = ax^4 + bx^3 + cx^2 + dx + e$, then:

$$\text{Sum of roots} = \sum \alpha = -\frac{b}{a} \quad (10)$$

$$\text{Sum of roots taken two at a time} = \sum \alpha\beta = \frac{c}{a} \quad (11)$$

$$\text{Sum of roots taken three at a time} = \sum \alpha\beta\gamma = -\frac{d}{a} \quad (12)$$

$$\text{Product of roots} = \prod \alpha = \frac{e}{a} \quad (13)$$

Polynomials of degree n

If $\alpha_1, \alpha_2, \dots, \alpha_n$ are the roots of $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, then:

$$\text{Sum of roots} = \sum \alpha = -\frac{a_{n-1}}{a_n} \quad (14)$$

$$\text{Sum of roots taken two at a time} = \sum_{i,j,i \neq j} \alpha_i \alpha_j = \frac{a_{n-2}}{a_n} \quad (15)$$

$$\text{Sum of roots taken } k \text{ at a time} = (-1)^k \frac{a_{n-k}}{a_n} \quad (16)$$

$$\text{Product of roots} = \prod \alpha = (-1)^n \frac{a_0}{a_n} \quad (17)$$

Polynomial Division

General Form

A polynomial can be expressed in quotient form:

$$P(x) = D(x) \cdot Q(x) + R(x) \quad (18)$$

or alternatively:

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)} \quad (19)$$

where $D(x)$ is the divisor, $Q(x)$ is the quotient and $R(x)$ is the remainder.

Note that the degree of the remainder is strictly less than the degree of the divisor:

$$\deg(R) < \deg(D) \quad (20)$$

Long Division

1. Any terms with a power less than the degree of the polynomial must be explicitly written down with a 0 coefficient.
2. Start from the left. Work out what term, when multiplied with the first term of the divisor, would give you the first term of the polynomial. Write this term above the line.
3. Then multiply the term you got in the previous step by the divisor and write that underneath the polynomial. Then subtract the relevant terms.
4. Repeat, this time starting with the second term of the polynomial, until the powers of the terms of the polynomial become less than the degree of the divisor.

Remainder Theorem

If a polynomial $P(x)$ is divided by $(x - a)$, the remainder is $P(a)$.

Proof. The polynomial can be expressed in the form:

$$P(x) = (x - a)Q(x) + R(x)$$

where $Q(x)$ is the quotient and $R(x)$ is the remainder.

Evaluating the polynomial at $x = a$ yields:

$$P(a) = (a - a)Q(a) + R(a) = R(a) \quad \square$$

Factor Theorem

If $(x - a)$ is a factor of $P(x)$, then $P(a) = 0$.

It follows as a corollary of the Remainder Theorem. This theorem can be used instead of polynomial long division when guessing and checking roots of a polynomial.

Proof. The remainder when $P(x)$ is divided by $(x - a)$ is zero, since $(x - a)$ is a factor. Hence, by the Remainder Theorem, $P(a) = 0$. □

The reverse is also true:

If $P(a) = 0$, then $(x - a)$ is a factor of $P(x)$.

Proof. By the Fundamental Theorem of Algebra, $P(x) = (x - x_1)(x - x_2) \dots (x - x_n)$. Proof by contradiction. Suppose that $(x - a)$ is not a factor of $P(x)$. Hence, for all $1 \leq i \leq n$, $x_i \neq a$ and $(a - x_i) \neq 0$, which means that $P(a) \neq 0$ since it is a product of non-zero numbers. This contradicts the fact that $P(a) = 0$, and therefore $(x - a)$ is a factor of $P(x)$. \square

Graphing Polynomials

In general, when you graph polynomials, consider its x -intercepts, y -intercept, stationary points and behaviour as $x \rightarrow \pm\infty$:

1. Find the x -intercepts by finding the real solutions of $P(x) = 0$. The multiplicity of the root (the number of times a root is repeated) will determine the shape of the graph near the root. For example, a double root will have a similar shape to $y = x^2$ (it will touch but not cross the x -axis), and a triple root will have a similar shape to $y = x^3$ (it will have a kink at the x -axis).
2. Find the y -intercept by finding $P(0)$.
3. Find the stationary points by solving $P'(x) = 0$, and their nature (local minimum or maximum) by using $P''(x) = 0$ or by testing the value of $P'(x)$ in the vicinity of the stationary points. Note that:
 - If the degree of the polynomial is even, the number of stationary points is some odd number that is less than the degree.
 - If the degree of the polynomial is odd, the number of stationary points is some even number that is less than the degree.
4. Consider the behaviour of the graph towards the far left and right:
 - If the degree of the polynomial is even:
 - If the leading coefficient is positive, as $x \rightarrow \pm\infty$, $P(x) \rightarrow \infty$.
 - If the leading coefficient is negative, as $x \rightarrow \pm\infty$, $P(x) \rightarrow -\infty$.
 - If the degree of the polynomial is odd:
 - If the leading coefficient is positive, as $x \rightarrow \infty$, $P(x) \rightarrow \infty$, and as $x \rightarrow -\infty$, $P(x) \rightarrow -\infty$.
 - If the leading coefficient is negative, as $x \rightarrow \infty$, $P(x) \rightarrow -\infty$, and as $x \rightarrow -\infty$, $P(x) \rightarrow \infty$.