# Curve Sketching and Polynomials

## The Equations of Circles

The equation of a circle, centre (0,0) and radius r units, is
$x^2 + y^2 = r^2$.

The equation of a circle, centre (h, k) and radius r units, is
$(x - h)^2 + (y - k)^2 = r^2$.

## Polynomials

A **polynomial** in the variable x, denoted by P(x), is expressed as follows:
$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + ... + a_2 x^2 + a_1 x + a_0$ where:

- The powers of x are positive integers
- $a_n, a_{n-1}, a_{n-2}, ... a_1$ are called (real number) coefficients
- n (the highest power) is called the **degree of the polynomial**
- $a_n x^n$ is the **leading term**
- $a_n$ is the **leading coefficient**
- $a_0$ is the **constant term** (independent of x)

Furthermore:
- If $a_n = 1$, P(x) is called a **monic polynomial**
- If n = 1, the polynomial is **linear**
- If n = 2, the polynomial is a **quadratic**
- If n = 3, the polynomial is a **cubic**
- If n = 4, the polynomial is a **quartic**
- If n = 5, the polynomial is a **quintic**
- P(c) is the value of P(x) at x = c

Polynomial expressions can be added, subtracted and multiplied.

## The remainder theorem

If a polynomial P(x) is divided by (x – a) then the remainder is P(a).

## The factor theorem

If a polynomial P(x) is divided by (x – a) such that the remainder is zero, then (x – a) is a factor of P(x).

## Sketching polynomials

The graph of the equation $y = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + ... + a_2 x^2 + a_1 x + a_0$ is called a **polynomial function**.

The zeros of $P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + ... + a_2 x^2 + a_1 x + a_0$ are the roots of y = P(x). Graphically, it is where the curve cuts the x-axis.

## Sketching polynomials with repeated roots

If $(x - a)^2$ is a factor of a polynomial P(x) then $P(x) = (x - a)^2Q(x)$.
We say P(x) has a double root at x = a.

The graph of y = P(x) will touch the x-axis at x = a
(ie. take the shape of a parabola at x = a).

If $(x - a)^3$ is a factor of a polynomial P(x) then $P(x) = (x - a)^3Q(x)$.
We say P(x) has a triple root at x = a.

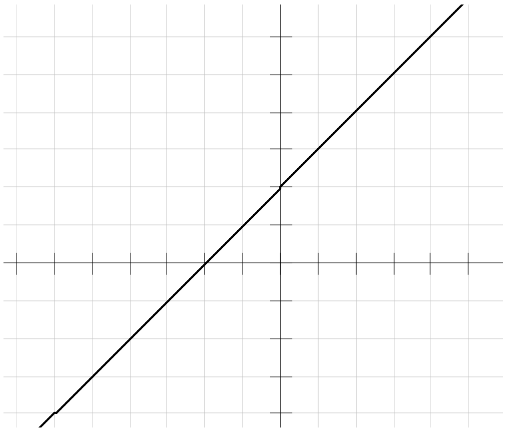
The graph of y = P(x) will cut the x-axis at x = a
(ie. take a shape similar to $y = x^3$ at x = a).

## Summary of different graphs

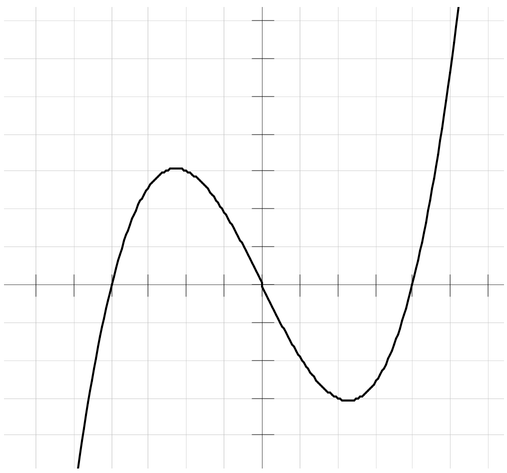
Consider $P(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_2x^2 + a_1x + a_0$

**For n odd and $a_n$ positive:**

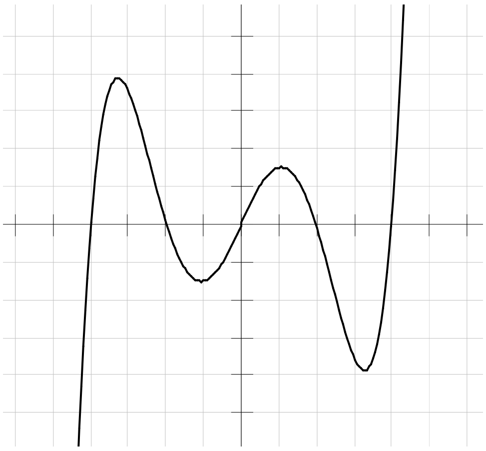
a) n = 1



b) n = 3

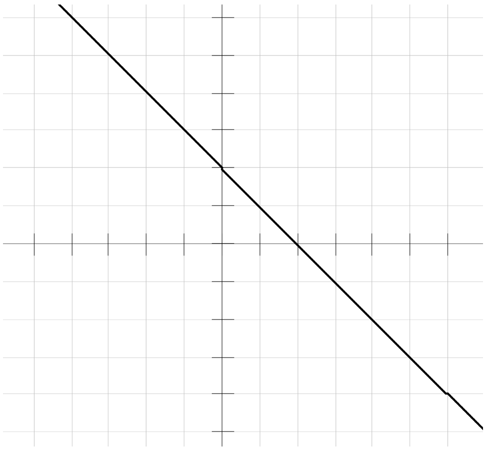


c) n = 5

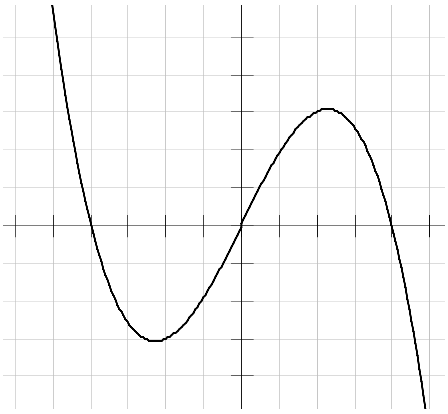


**$a_n$ negative:**

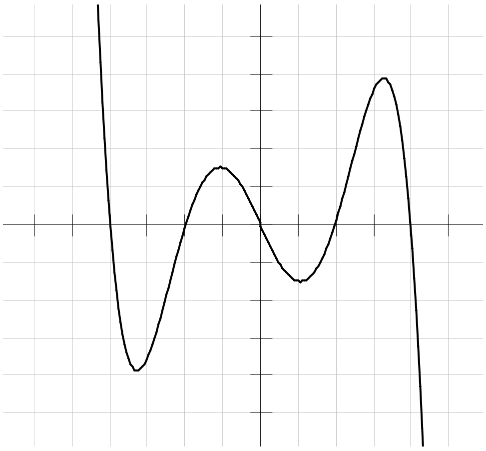
a) n = 1



b) n = 3

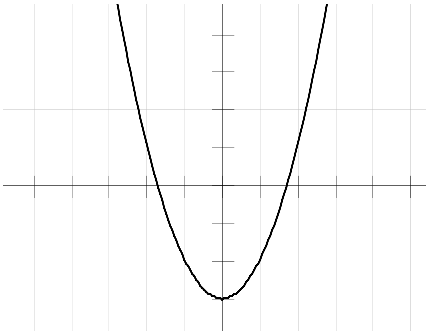


c) n = 5

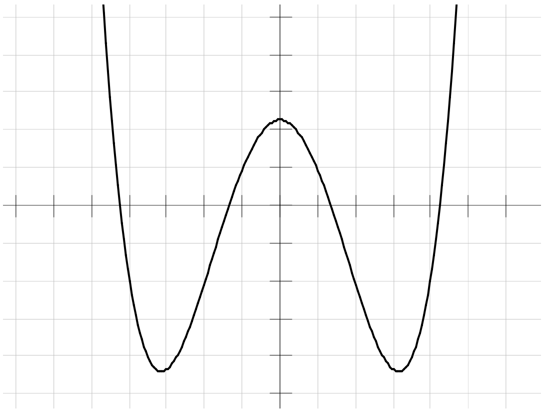


**For n even, $a_n$ positive:**

a) n = 2

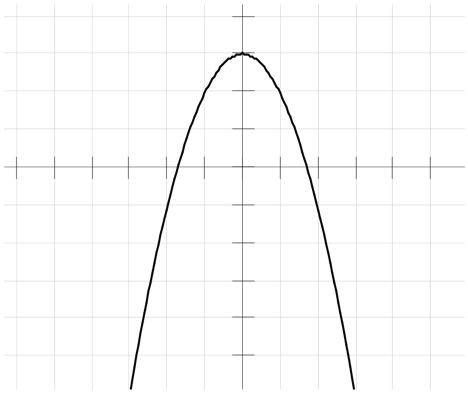


b) n = 4

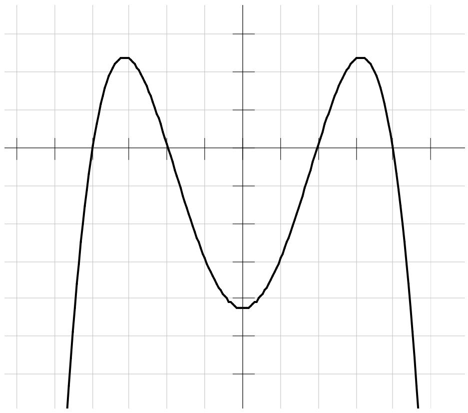


**$a_n$ negative:**

a) n = 2



b) n = 4



# Functions and Logarithms

## Variables

When we look at how two quantities are related, the quantities are referred to as **variables** because they change or vary. The numbers they take are called their **values**.

In situations where there are two variables, the variables may be either independent or dependent. For example, the **independent variable** x may have any value, but the **dependent variable** y takes values that are dependent on x.

The variables are graphed on different axes. The independent variable is always shown on the horizontal axis and the dependent variable on the vertical axis.

## Functions

A **function** is defined as a relationship between variables where, for each value of the independent variable, there exists only one value of the dependent variable.

If x is the independent variable and y is the dependent variable, a function is defined as follows:
A function is a relationship where for each x-value there is only one y-value.

## Vertical Line Test

Any vertical line will cut the graph of a function only once.

## Function notation

If y is a function of x, we may write y = $f$(x). the expression $f$(2) means the value of the function when x = 2 (or the y-value when x = 2).

## Graphs of functions

A function can be represented by:
- Ordered pairs (or a table of values)
- A formula
- A graph

The **domain of the function** is the set of possible values for the independent variable.
The **range of the function** is the set of possible values for the dependent variable.

## Inverse functions

The **inverse function** is the function obtained when the dependent and independent variables of some functions are interchanged.

The inverse function to $f(x)$ is written as $f^{-1}(x)$. The graph of $y = f(x)$ is always a reflection of the graph of $y = f^{-1}(x)$ in the line $y = x$.

Functions that have inverse functions are those functions where there is only one x-value for each y-value, ie. They pass the horizontal line test.

The horizontal line test indicates whether or not a function has an inverse (The vertical line test indicates whether a relation is a function).

## Logarithms

The logarithm of a number N is the index b to which the base a must be raised to give the number N.
So, **if $N = a^b$ then $\log_a N = b$**
(where N is the number, a is the base and b is the index or power).

## The Laws of Logarithms

**Law 1**

$$\log_a (xy) = \log_a x + \log_a y$$

**Law 2**

$$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$$

**Law 3**

$$\log_a x^n = n \log_a x$$

**Law 4**
The logarithm of 1 to any base is equal to 0:

$$\log_a 1 = 0$$

**Law 5**
The logarithm of any number *a* to base *a* equals 1:

$$\log_a a = 1$$

**Law 6**

$$\log_a \left(\frac{1}{x}\right) = -\log_a x$$

**Law 7**

$$\log_a a^m = m$$

# The graph of $y = a^x$ and $y = \log_a x$, $a > 0$

Since $y = \log_a x$ is the inverse of $y = a^x$, the graph of $y = \log_a x$ can be drawn by reflecting $y = a^x$ in the line $y = x$.