Quadratic expressions and equations

Expressions such as $x^2 + 3x$, $a^2 – 7$ and $4t^2 – 9t + 5$ are called **quadratic expressions** because the highest power of the variable is 2. The word comes from the Latin *quadratus* meaning square or squared.

Expressions such as $m^2 – 3 = 0$, $2x^2 – 5x – 3 = 0$ and $5g^2 – 13g = 0$ are called **quadratic equations** since the highest power of the variable is also 2.

A **quadratic relationship** is an equation that relates two variables. The power of the dependent variable is 1 while the highest power of the other variable is 2. We can therefore say that $h = 4t^2 – 3t$ is a quadratic relationship where the power of h is 1 and the highest power of t is 2. Other examples of quadratic relationships are $d = 9t^2 – 5$, $y = 3x^2 + 4x – 1$ and $A = \pi r^2$.

When a quadratic expression is equal to zero, the equation is called a quadratic equation.

The parabola $y = x^2$

The equation $y = x^2$ is a quadratic relationship (or quadratic equation). The graph of the equation $y = x^2$ is a smooth curve called a **parabola**.

Features of $y = x^2$:

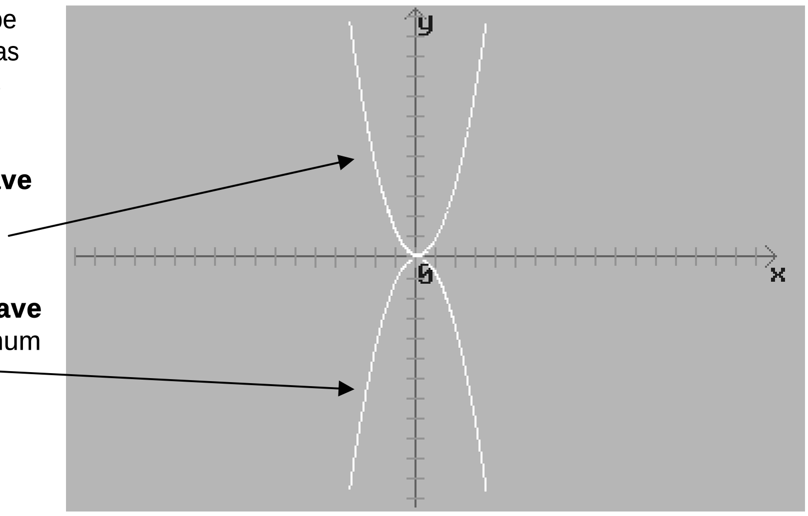
- The graph is symmetrical about the y-axis.
- The axis of symmetry is also called the **axis of the parabola**. For $y = x^2$, the equation of the axis of symmetry is $x = 0$ (the equation of the y-axis).
- The point of intersection of a parabola with its axis (or with the axis of symmetry) is the **vertex**. For $y = x^2$, the vertex is the point (0, 0). The vertex is also called the **turning point**, since the parabola changes direction at this point.
- The y-intercept is the point where the parabola cuts the y-axis. For $y = x^2$, the y-intercept is 0.
- The x-intercept if 0 (where the curve cuts the x-axis).
- The minimum value of $y = x^2$ is 0.

Concavity

Concavity refers to the shape of a curve or curved surface as seen from the inside (such as the inside of a hollow ball).

The parabola $y = x^2$ is **concave up**, and so it has a minimum turning point.

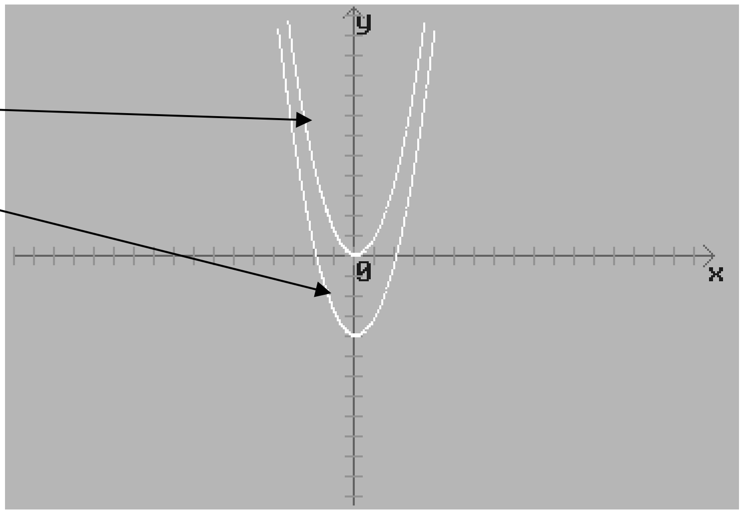
The parabola $y = -x^2$ is **concave down**, and so it has a maximum turning point.



In general, if the coefficient of x^2 is positive, the parabola is concave up, and if the coefficient of x^2 is negative, the parabola is concave down.

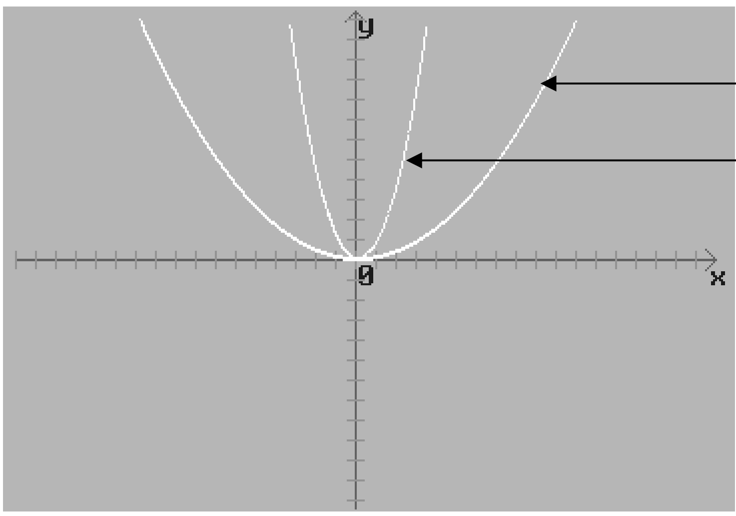
The effect of the constant term

The effect of the constant term c on the graph of $y = \pm x^2 + c$ is to move the parabola up or down.



The effect of the coefficient of x^2

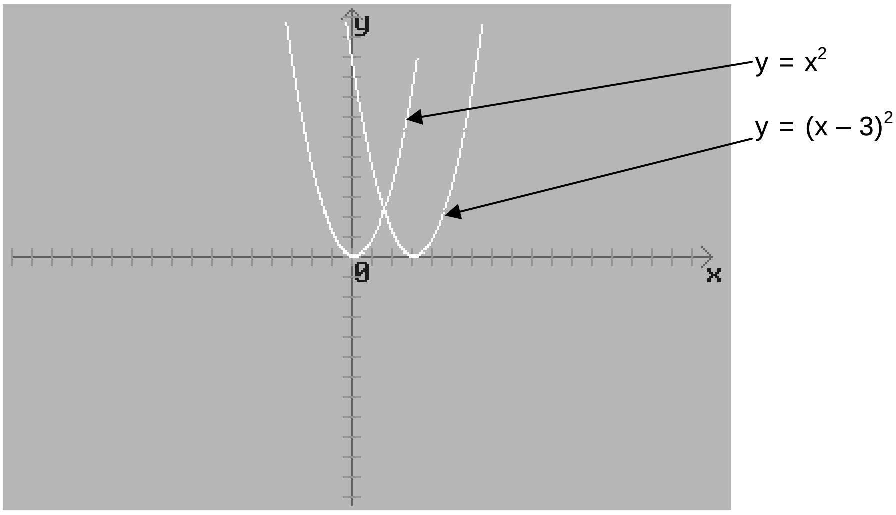
For the graph of $y = ax^2$ the size of a (the coefficient of x^2) determines whether the parabola is 'wide' or 'narrow'.



As the size of a increases, the graph of the parabola becomes 'narrower' and as the size of a decreases, the graph of the parabola 'widens'.

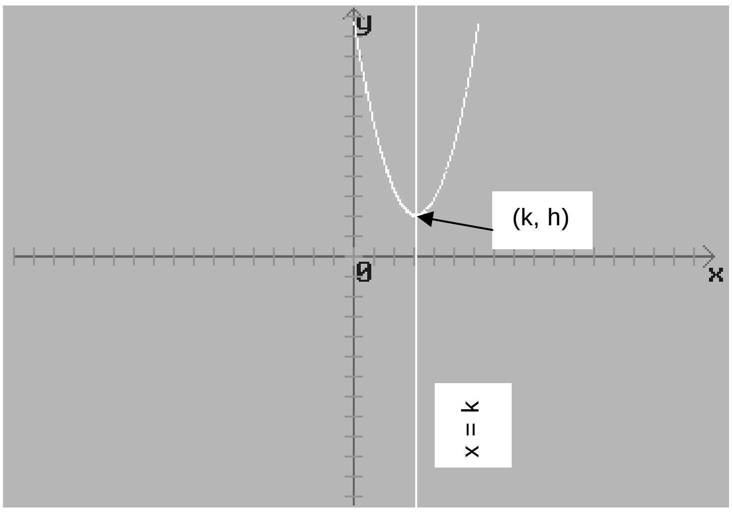
The parabola $y = (x - k)^2$

The effect of the constant k on the graph of $y = (x - k)^2$ is to move the parabola to the right if k is positive and to the left if k is negative.



The graph of $y = a(x - k)^2 + h$

The graph of the equation $y = a(x - k)^2 + h$ is a parabola with vertex (k, h). If $a > 0$ the parabola is concave up, and if $a < 0$ the curve is concave down.



$y = a(x - k)^2 + h$, $a > 0$

The equation of the axis of symmetry is $x = k$ and the minimum or maximum value of y is h.

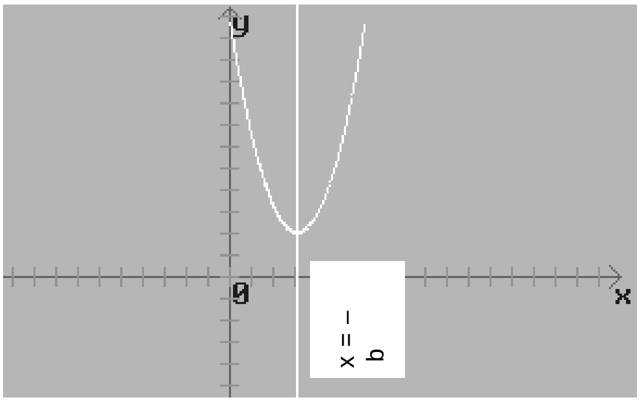
The general form of the quadratic relationship

The equation $y = a(x - k)^2 + h$ is sometimes called the **vertex form** of the quadratic relationship of parabola. This equation can be expanded and simplified to give the general form of the quadratic relationship.

> The general form of the quadratic relationship is $y = ax^2 + bx + c$ where a, b and c are real numbers and $a \neq 0$. The graph of this relationship is a parabola.

The axis of symmetry of a parabola

> If the equation of a parabola is in the general form $y = ax^2 + bx + c$, the equation of its axis of symmetry is $x = -\frac{b}{2a}$.



$a > 0$

The minimum or maximum value os found by substituting $x = -\frac{b}{2a}$ in the equation of the parabola.

Solving quadratic equations

The **general form** of a quadratic relationship of a parabola is $y = ax^2 + bx + c$.

The intersection of a parabola and a straight line

The number of solutions to a quadratic equation is the same as the number of points of intersection between the corresponding parabola and straight line.

<u>Example:</u>
To solve $x^2 - 6x + 5 = 0$, we can rewrite the equation as $x^2 = 6x - 5$ and then draw the graphs of $y = x^2$ and $y = 6x - 5$.

The intersection of a parabola and the x-axis

The number of real solutions to a quadratic equation is the same as the number of intercepts the corresponding parabola has with the x-axis.

Solving quadratic equations by factorising

When solving quadratic equations by factorising, the following property is used:

> If pq = 0 where p and q are any real numbers, then p = 0 or q = 0 or p = q = 0.

> Solving quadratic equations by factorising involves three steps:
> 1. Arrange the quadratic equation in the form $ax^2 + bx + c = 0$.
> 2. Factorise (into a product of two terms).
> 3. Use the property that if pq = 0, then p = 0 or q = 0.

Solving quadratic equations by completing the square

Quadratic equations can be solved by using the method called 'completing the square'. (This method can also be used to find the vertex or turning point of a parabola.)

The method of completing the square depends on the following results for perfect squares:

> $(x + a)^2 = x^2 + 2ax + a^2$
> $(x – a)^2 = x^2 – 2ax + a^2$

<u>Example:</u> Solve $x^2 + 6x + 7$

1. Move the constant term to the right-hand side.
   $\therefore x^2 + 6x = –7$
2. Divide by the coefficient of x^2.
   $x^2 + 6x = –7$
3. Halve the coefficient of x, square it and then add the square to both sides.
   $x^2 + 6x + 3^2 = –7 + 3^2$
   $\therefore x^2 + 6x + 9 = 2$
4. Express the LHS as a perfect square.
   $(x + 3)^2 = 2$
5. Solve the resulting equation.
   $x + 3 = \pm\sqrt{2}$ (taking the square root of both sides)
   $x = –3 \pm \sqrt{2}$
   $\therefore x = –3 + \sqrt{2}$ or $x = –3 – \sqrt{2}$
   $\therefore x \approx –1.59$ or $x \approx –4.41$ (expressing the answer correct to 2 decimal places)

<u>Example:</u> Find the coordinates of the vertex of the parabola with equation $y = 2x^2 + 4x – 7$ using the method of completing the square.

$$\begin{aligned} y &= 2x^2 + 4x - 7 \\ &= 2\left(x^2 + 2x - \frac{7}{2}\right) \\ &= 2\left((x^2 + 2x + 1) - 1 - \frac{7}{2}\right) \\ &= 2\left((x + 1)^2 - \frac{9}{2}\right) \end{aligned}$$

$= 2(x + 1)^2 - 2 \times \frac{9}{2}$

$\therefore y = 2(x + 1)^2 - 9$

$\therefore$ Coordinates of the vertex are (–1, –9).

Solving quadratic equations by using the quadratic formula

The quadratic formula for $ax^2 + bx + c = 0$ is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Graphs

The world is always changing and we are constantly trying to find models that will describe the change. One way of describing this change is by using graphs.

A mathematical model is a device (such as a formula) that describes, in mathematical terms, some natural phenomenon. This model is then used to produce information about that phenomenon.

Some of the many areas in which modelling is applied include traffic flow, the flow of money in the economy, the effectiveness of medical treatments, space travel, electrical networks and weather forecasting.

Modelling: A way of describing change

We can use mathematical models to describe change. When we try to apply mathematics to the real world, we often produce a mathematical description of some kind to help solve practical problems. This technique is called **mathematical modelling**. This mathematical description may be a formula that expresses the relationship between two variables, or it may be a graph which illustrates (or approximates) the relationship.

The model can be a:
- Formula
- Table of values
- Graph

Variables

When using graphs to show how two quantities are related, the quantities are referred to as **variables** because they change or vary.

Speed graphs

Speed graphs are line graphs that show how speed and time are related.

Features of speed graphs:
- If the line in the graph is horizontal, the speed is constant.
- If the line rises with time, speed is increasing.
- If the line falls with time, speed is decreasing.

The straight line

The general form of a linear equation is $ax + by + c = 0$, where a, b and c are integers. The gradient intercept form of a linear equation is $y = mx + b$ where m is the gradient and b is the y-intercept. The graph of a linear equation is a straight line.

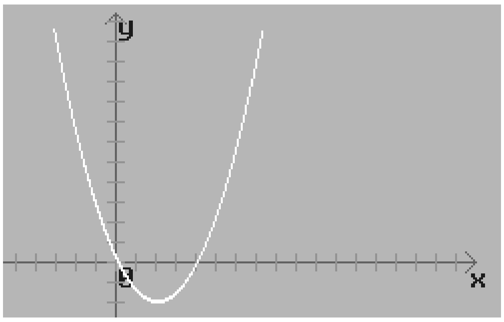
Lines of best fit

Graphs are often drawn using the results from practical situations and experiments. The line that best describes the results is called the **line of best fit**.

The parabola

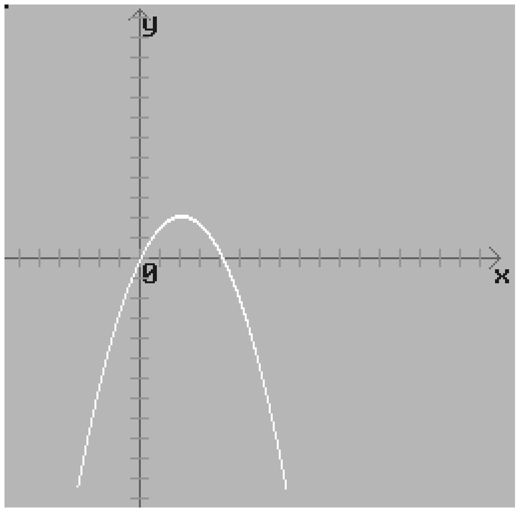
The general form of the equation of a parabola is $y = ax^2 + bx + c$ where a, b and c are real numbers and $a \neq 0$.

The equation of the parabola may also be expressed in the form $y = a(x - k)^2 + h$ where (k, h) is the vertex of the parabola.



$y = ax^2 + bx + c$

$a > 0$



$y = ax^2 + bx + c$

$a < 0$

Exponential graphs

> The general exponential equation is:
> $y = a^x$ where $a > 0$.
> x is called the exponent (power) of a in the expression a^x.

Features of exponential graphs:

- The y-intercept is 1 since $a^0 = 1$.
- As x approaches a large negative number, a^x approaches zero. This means the graph of $y = a^x$ approaches the x-axis as x approaches a large nagative number. We usually write: 'As $x \longrightarrow -\infty$, $a^x \longrightarrow 0$'.
- As x approaches a large positive number, a^x becomes very large. This is written: 'As $x \longrightarrow \infty$, $a^x \longrightarrow \infty$'. Graphically, this means the graph of $y = a^x$ increases very rapidly.

In the graph $y = a^x + k$, k moves the graph up if positive and the graph down if it negative.
In the graph $y = b \times a^x$, where $b > 0$, the y-intercept of the graph $y = b \times a^x$ is b; also the graph increases more rapidly if $b > 1$ and less rapidly if $0 < b < 1$.

The reflection of $y = a^x$ in the y-axis is $y = a^{-x}$.
The reflection of $y = a^x$ in the x-axis is $y = -a^x$.

When $y = a^{(x-b)}$,
For $b > 0$, the exponential graph slides b units to the right along x-axis.
For $b < 0$, the exponential graph slides b units to the left.

Asymptotes

A line that a graph approaches but never touches is called an **asymptote**.

Hyperbola

If y varies inversely with x,

$$y \propto \frac{1}{x}$$

$\therefore y = \frac{k}{x}$, where k is called the **constant of variation**.

This equation may also be written as $xy = k$.
The graph of the equation $y = \frac{k}{x}$ is called a **hyperbola**.

> The general equation for the graph of a hyperbola is:
> $y = \frac{k}{x}$, $k \neq 0$.

Features of the hyperbola:

- The graph of $y = \frac{k}{x}$ is discontinuous at $x = 0$.
- The graph has 2 parts or branches.
- The graph as two axes of symmetry. Their equations are $y = x$ and $y = -x$.
- The graph has 2 asymptotes, which are the x-axis and the y-axis.
- The equation may also be written as $xy = k$, $k \neq 0$.

In $y = \frac{1}{x} + 1$, the + 1 moves $y = \frac{1}{x}$ 1 unit up the y axis.
In $y = \frac{1}{x-1}$, $y = \frac{1}{x}$ is moved 1 unit to the right along the y-axis.

Equation of the circle

The equation of the circle, centre (p, q) radius r is:
$(x - p)^2 + (y - q)^2 = r^2$

The equation of a circle, centre (0, 0) and radius r units, is:
$x^2 + y^2 = r^2$

The graph of $y = \sqrt{r^2 - x^2}$ is a semicircle above the x-axis.
The graph of $y = -\sqrt{r^2 - x^2}$ is a semicircle below the x-axis.