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Maths Corner 2: Infinity ∞

What does infinite and infinity mean? What are the "rules" for operations with infinity or "transfinite numbers"? How many "infinities" are there?

Firstly, what does the dictionary say? According to the *Collins Dictionary of Mathematics*, an *infinite* number is "not finite" (pretty obvious, hey?), and it has "a size or absolute value that is greater than any natural number." When a set is called infinite, it means that it can be "put into a one-to-one correspondence with a proper subset of itself" (more on this later). Thus, *infinity* is a value *greater than any computable value*. Just to give an example, the numbers 1, 2, 3… constitute an infinite collection. (Note: infinite is an adjective, while infinity is a noun.)

In a series of papers published between 1874 and 1884, George Cantor showed that infinity could be treated mathematically by defining an infinite number to be one that could be put into a one-to-one correspondence with some part of itself. We'll show that the number of positive integers (1, 2, 3…) is equal to the number of even integers (2, 4, 6…) by writing them visually like this:

1	2	3	4	5	...
⇕	⇕	⇕	⇕	⇕	
2	4	6	8	10	...

In the same way, we can show that the number of integers is the same as the number of odd numbers. (When Galileo saw this, he concluded that infinity was "inherently incomprehensible"!)

1	2	3	4	5	...
⇕	⇕	⇕	⇕	⇕	
1	4	9	16	25	...

We paired each of these sets in a *systematic manner*, in a one-to-one correspondence. Cantor in fact found some very surprising results. He discovered that the number of points on a line is the same as the number of points in a plane, and it is in fact equal to the number of points in a space of any dimension (contrary to what was believed at that time).

However, the important thing to remember is that *not all infinite numbers are equal*. Cantor demonstrated that positive integers 1, 2, 3… could not be put into a one-to-one correspondence with the number of points on a line. Even though both were *infinite*, the number of points on a line was *larger* than the number of positive integers.

So, how many infinite numbers are there? No surprises here: there are an infinite number of infinite numbers! Cantor assigned the symbol $\aleph_0$ (pronounced aleph-null) to the smallest infinity, the one represented by the positive integers (which also represents the set of squares, as we showed above, among other things). The next larger infinite number is $\aleph_1$, aleph-one, and so on. Cantor called the alephs *transfinite numbers*.

Some of you might say that 1 divided by 0 is infinity, but what we really mean here is that as x tends to 0, $1/x$ increases without bound (check the graph of $y = 1/x$: $x = 0$ is an asymptote). As well, parallel lines are said to *meet at infinity*. Such is the strange of world of infinity!

But can infinity be treated like a "normal" number, with addition and subtraction and so on? Well, say we have a hotel with an infinite number of rooms, all filled. A new guest arrives; we simply move the person in room 1 to room 2, the person in room 2 to room 3 and so on, and everyone can still fit. Even though we added 1 to the number of guests, we still have an *infinite* number of guests! With the operations of arithmetic, we can say that infinity remains… infinity! (The other operations work in about the same way.)

Primary reference: Morris, Richard. *Achilles in the Quantum Universe: The Definite History of Infinity*