Enoch Lau

Maths Corner **Problem 1 Stick Tree**

a) What is the total length of all the branches on any given day?

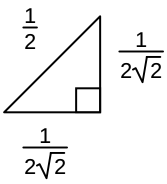
Let's consider the case for:
Day 1: The length of the single branch is 1m.
Day 2: Because there are 2 branches of length $\frac{1}{2}$ m, the sum of the lengths of the branches is $1 + \frac{1}{2} + \frac{1}{2} = 1 + \frac{2}{2} = 2$m.
Day 3: 4 smaller branches of length $\frac{1}{4}$ m have grown on, so the sum of the lengths of the branches is $1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{2}{2} + \frac{4}{4} = 3$m.
Clearly, for Day n, the sum of the lengths of the branches is n metres.

Formal proof:
For the trivial case (Day 1), the length is 1m. We observe that the number of branches added on Day n is 2^{n-1}.
Proof by induction: the statement is clear for $n = 1$ because $2^0 = 1$ branch on Day 1. Assume that the number of branches present on Day k is 2^{k-1}. Because each of those branches that grew on Day k splits into two, the number of branches in Day (k + 1) is $2 \times 2^{k-1} = 2^k$, which is of the required form. Hence, for all positive integers n, the number of new branches is 2^{n-1}.

However, those 2^{n-1} branches are each of length $\frac{1}{2^{n-1}}$ m, because the length of each new branch is half that of the preceding layer, and the length on Day 1 is 1m. Hence, the additional branch length added each day is $2^{n-1} \times \frac{1}{2^{n-1}} = 1$m.
Therefore, the total length of all the branches on any given Day n, is $\underbrace{1 + 1 + \ldots + 1}_{n \text{ terms}} = n$ metres.

b) What is the maximum height and width that the tree will grow to?
The height on Day 1 is 1m, the height added on Day 2 is $\frac{1}{2\sqrt{2}}$ m (by Pythagoras' Theorem*), the height added on Day 3 is $\frac{1}{4}$ m, the height added on Day 4 is $\frac{1}{8\sqrt{2}}$ m, and so it goes on.

* Pythagoras' Theorem:



$$\left(\frac{1}{2\sqrt{2}}\right)^2 + \left(\frac{1}{2\sqrt{2}}\right)^2 = \frac{1}{2}$$

If you fiddle on your calculator, you'll get a value somewhere above 1.8m or thereabouts.

Working out the maximum height is pretty similar, only that you don't add the 1m original branch and you have to multiply the result by two because the tree has two sides.

Formal proof:
We can consider the problem in two stages, one considering all of the vertical branches and one considering all of the sloping branches.

All the vertical branches (we skip every second one because every second one is inclined to the horizontal): 1m, $\frac{1}{4}$ m, $\frac{1}{16}$ m, $\frac{1}{32}$ m, etc… Each successive branch in this series is reduced by a factor of 4.
This is a geometric series, $a = 1$, $r = \frac{1}{4}$, so:

$$S_\infty = \frac{a}{1-r} = \frac{1}{1-\left(\frac{1}{4}\right)} = \frac{4}{3}$$

For the first sloping branch (see previous diagram), the height is $\frac{1}{2\sqrt{2}}$ m. Because we skip every second one, each successive term in this series will be reduced by a factor of 4.
This produces a geometric series, $a = \frac{1}{2\sqrt{2}}$, $r = \frac{1}{4}$, so:

$$S_\infty = \frac{a}{1-r} = \frac{\left(\frac{1}{2\sqrt{2}}\right)}{1-\left(\frac{1}{4}\right)} = \frac{1}{2\sqrt{2}} \times \frac{4}{3} = \frac{\sqrt{2}}{3}$$

Hence, the maximum height of the tree = $\frac{4}{3} + \frac{\sqrt{2}}{3} = \frac{4+\sqrt{2}}{3}$ metres (or about 1.80m).

Now, considering only the right hand side of the tree, the horizontal components can be divided into those belonging to branches that are horizontal and those belonging to branches at an inclined angle. Using a method similar to the one above,

Horizontal branches: geometric series $a = \frac{1}{4}$, $r = \frac{1}{4}$ (again, we skip every second one):

$$S_\infty = \frac{a}{1-r} = \frac{\left(\frac{1}{4}\right)}{1-\left(\frac{1}{4}\right)} = \frac{1}{3}$$

Branches at an inclined angle: geometric series $a = \frac{1}{2\sqrt{2}}$, $r = \frac{1}{4}$:

$$S_\infty = \frac{a}{1-r} = \frac{\left(\frac{1}{2\sqrt{2}}\right)}{1-\left(\frac{1}{4}\right)} = \frac{1}{2\sqrt{2}} \times \frac{4}{3} = \frac{\sqrt{2}}{3}$$

Hence, the maximum width of the tree is $2 \times \left(\frac{1}{3} + \frac{\sqrt{2}}{3}\right) = \frac{2+2\sqrt{2}}{3}$ metres (or about 1.61m).