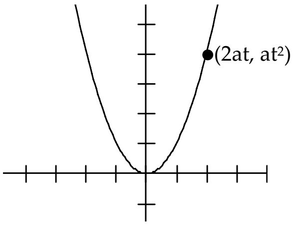
# Parametric Equations and the Parabola (Mathematics Extension 1)

$$\boxed{x = 2at \qquad y = at^2}$$

## Changing Parametric Equations to Cartesian Equations

### Parabolas ($x = 2at$, $y = at^2$):



$x = 2at$

$$t = \frac{x}{2a} \quad (1)$$

$y = at^2$

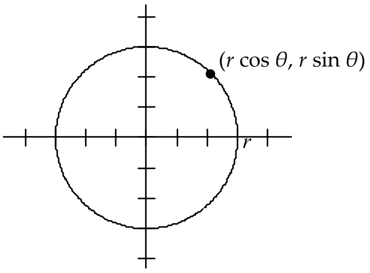
Sub in (1):

$$y = a\left(\frac{x}{2a}\right)^2 = \frac{x^2}{4a}$$

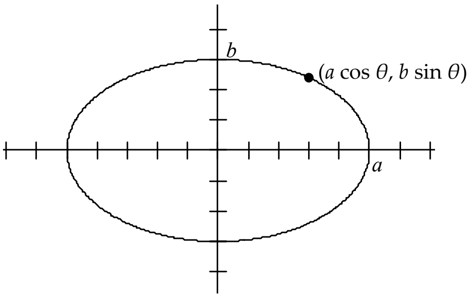
$\Rightarrow x^2 = 4ay$, which is the equation of the parabola, vertex (0, 0), focus (0, $a$).

### Circles ($x = r\cos\theta$, $y = r\sin\theta$):

$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$, which is the equation of the circle centre (0, 0), radius $r$.



### Ellipses ($x = a\cos\theta$, $r\sin\theta$):



$$x^2 = a^2\cos^2\theta$$

$$\therefore x^2 = a^2(1 - \sin^2\theta)$$

$$y^2 = b^2\sin^2\theta$$

$$\Rightarrow \sin^2\theta = \frac{y^2}{b^2}$$

$$\therefore x^2 = a^2\left(1 - \frac{y^2}{b^2}\right)$$

$$\Rightarrow \frac{x^2}{a^2} = 1 - \frac{y^2}{b^2}$$

$\Rightarrow \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, which is the equation of an ellipse centred at the origin.

## Parametric Equation of Tangent to $x^2 = 4ay$

Point $P$ $(2ap, ap)$

**1.** Find gradient

$$y = \frac{x^2}{4a} \Rightarrow y' = \frac{x}{2a}$$

When $x = 2ap$, $y' = \dfrac{2ap}{2a} = p$

**2.** Use $y - y_1 = m(x - x_1)$

$y - ap^2 = p(x - 2ap)$ → $\boxed{\therefore y = px - ap^2}$

## Parametric Equation of Normal

**1.** Find gradient

$$m = -\frac{1}{p}$$

**2.** Use $y - y_1 = m(x - x_1)$

$$y - ap^2 = -\frac{1}{p}(x - 2ap)$$

$$py - ap^3 = -x + 2ap$$

→ $\boxed{\therefore x + py = ap^3 + 2ap}$

## Intersection of Tangents and Normals

**1.** Find the point of intersection of the two tangents at $P\ (2ap,\ ap^2)$ and $Q\ (2aq,\ aq^2)$ by solving $y = px - ap^2$ and $y = qx - aq^2$.

$y = px - ap^2$ **(1)**
$y = qx - aq^2$ **(2)**

Sub **(1)** into **(2)**:

$$px - ap^2 = qx - aq^2$$

$$px - qx = ap^2 - aq^2$$

$$x(p-q) = a(p-q)(p+q)$$

$$\therefore x = a(p+q)$$

Sub this result into **(1)**:

$$\begin{aligned} y &= p(ap + aq) - ap^2 \\ &= apq \end{aligned}$$

→ $\boxed{\text{Intersection is at } (a(p+q), apq)}$

**2.** Find the point of intersection of the normals from $P\ (2ap,\ ap^2)$ and $Q\ (2aq,\ aq^2)$ by solving $x + py = ap^3 + 2ap$ and $x + qy = aq^3 + 2aq$.

$x + py = ap^3 + 2ap$ **(1)**

$x + qy = aq^3 + 2aq$ **(2)**

**(1)** – **(2)**:

$$py - qy = ap^3 - aq^3 + 2ap - 2aq$$

$$y(p-q) = a(p-q)(p^2 + pq + q^2) + 2a(p-q)$$

$$\therefore y = a(p^2 + pq + q^2 + 2)$$

Sub this result into **(2)**:

$$x + pa(p^2 + pq + q^2 + 2) = ap^3 + 2ap$$

$$x + 2ap + ap^3 + ap^2q + apq^2 = ap^3 + 2ap$$

$$x = -(ap^2q + apq^2)$$

$$\therefore x = -apq(p+q)$$

→ $\boxed{\begin{array}{l}\text{Intersection is at} \\ (-apq(p+q), a(p^2 + pq + q^2 + 2))\end{array}}$

## Focal Chords

**1.** Find the equation of the chord from $P$ $(2ap, ap^2)$ to $Q$ $(2aq, aq^2)$.

$$\frac{y-ap^2}{x-2ap}=\frac{aq^2-ap^2}{2aq-2ap}$$

$$(y-ap^2).2a(q-p)=(x-2ap).a(q-p)(q+p)$$

$$2(y-ap^2)=(x-2ap)(p+q)$$

$$2y-2ap^2=px+qx-2ap^2-2apq$$

$$y+apq=\frac{1}{2}x(p+q)$$

The equation of the chord is

$$\therefore y-\frac{1}{2}x(p+q)+apq=0$$

**2.** If $PQ$ is a focal chord, it goes through $(0, a)$.

From before, $y-\frac{1}{2}x(p+q)+apq=0$ is the equation of a chord.

Now, sub in $(0, a)$:

$$a-\frac{1}{2}.0.(p+q)+apq=0$$

$$a+apq=0$$

$$apq=-a$$

$$\Rightarrow pq=-1$$

**3.** Prove if $PQ$ is a focal chord, then their tangents will meet on the directrix.

From before, intersection of tangents is $(a(p+q), apq)$. Since $PQ$ is a focal chord, $pq=-1$. $\Rightarrow (a(p+q), -a)$. Because the directrix of the parabola $x^2=4ay$ is $y=-a$, the point of intersection of the tangents lies on the directrix.

## Chord of Contact

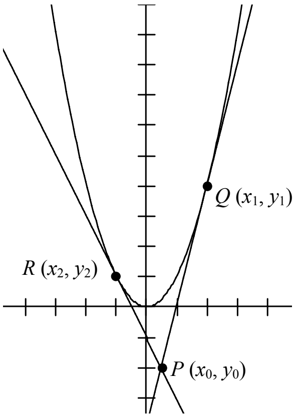
Let point of intersection of tangents be $P$ $(x_0, y_0)$.

If $Q$ is the point $(x_1, y_1)$, the tangent $PQ$ has equation

$$xx_1=2a(y+y_1)$$

Similarly, if $R$ is the point $(x_2, y_2)$, the tangent $PR$ has equation

$$xx_2=2a(y+y_2)$$



However, $P$ lies on both tangents, so:

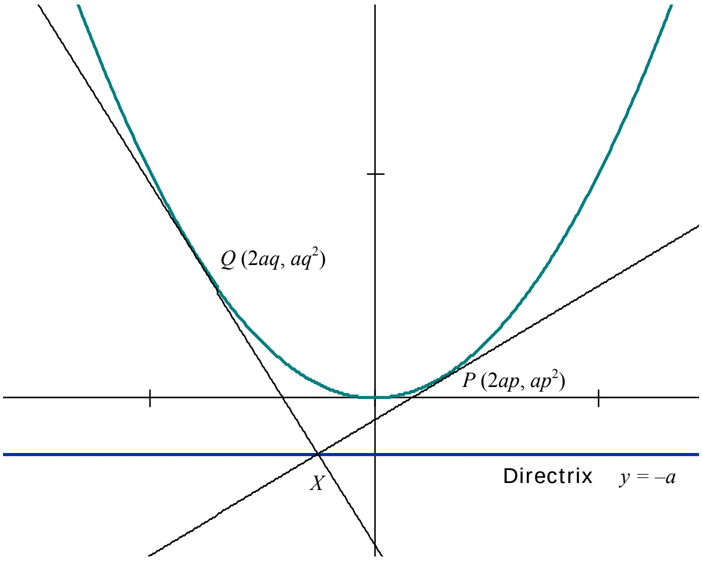
$PQ$: $x_0x_1=2a(y_0+y_1)$ **(1)**

$PR$: $x_0x_2=2a(y_0+y_2)$ **(2)**

From **(1)** and **(2)**, it is clear that $Q$ $(x_1, y_1)$ and $R$ $(x_2, y_2)$ lie on the line:

$$\boxed{xx_0=2a(y+y_0)}$$

## Properties of Parabolas



**Tangents at the end of a focal chord intersect at right angles on the directrix.**

From before, $X$, the point of intersection, is at $(a(p+q), apq)$.

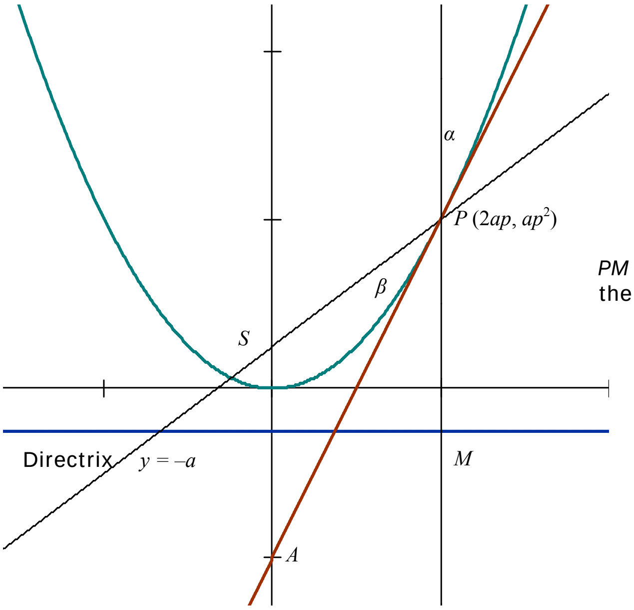
Since PQ is a focal chord, $pq = -1$.

$\therefore\ X\ (a(p+q), -a) \Rightarrow X$ lies on the directrix

Now, $m_{PX} = p$, $m_{QX} = q$

However, $pq = -1$. Hence, $PQ \perp QX$.

**Prove the angle of reflection is equal to the angle of incidence.**



*PM* is parallel to the *y*-axis

1. Equation of tangent: $y = px - ap^2$
2. Find point $A$ (sub $x = 0$):

$y = -ap^2 \qquad \Rightarrow A\ (0, -ap^2)$

3. Find distances. $AS = a + ap^2 \qquad SP = PM = a + ap^2$
4. $\therefore\ \Delta$ ASP is isosceles

$\therefore\ \angle SAP = \beta$.

However, $\angle SAP = \alpha$ (corresponding angles of parallel lines)

$\therefore\ \alpha = \beta$