COMP3610 Lecture Week 10

Enoch Lau

1 Vertex cover problem

- $G = (V, E)$
- Associated with each $v_i \in V$ is a weight w_i.
- Find a subset $S \subset V$ such that every edge $e \in E$ is incident on at least one $v \in S$ and minimise $\sum_{i \in S} w_i$.

2 Linear programming

- Standard form:
  - Minimise $c^T \cdot x$
  - Subject to $A \cdot x \geq b$, and $x \geq 0$
- Use the simplex algorithm (1948); even though it is not a polynomial time algorithm, in practice, it runs very quickly.
- Ellipsoid algorithm (1970's): polynomial time, but it wasn't very efficient; hence, this problem is in P
- Interior point algorithm (1980's)
- Integer programming problem: related to linear programming, but in addition to having $x \geq 0$, we have the constraint that x is an integer. This is an NP-hard problem.

3 Reducing vertex cover to integer programming

- Let $x_i = 1$ if $i \in S$, $x_i = 0$ otherwise.
- Minimise $\sum_{i \in V} w_i x_i$
- $x_i + x_j \geq 1 \, \forall \, e = (i, j) \in E$
- $x_i \in \{0, 1\} \, \forall \, i \in V$

- Typically, in a graph problem, the vertices become the x_i, choose the objective function and then the problem will give the constraints.
- Approximate the integer programming problem with a linear programming problem. Remove the constraint of $\{0, 1\}$. $w_{\text{LP}} \leq w_{\text{IP}}$, and thus we have a lower bound for the solution to the integer programming problem.
- To map the problem back, do rounding, i.e. $S = \{i \mid x_i \geq 0.5\}$. You still have a feasible solution, because if $x_i, x_j < 0.5$, then $x_i + x_j < 1$, which violates the constraint, and so you never lose vertices that should have been selected.
- $w_{\text{LP}} w \cdot x^* = \sum_i w_i x_i^* \geq \sum_{i \in S} w_i x_i^* \geq \frac{1}{2} \sum_{i \in S} w_i = \frac{1}{2} w(S)$
- $w(S) \leq 2w_{\text{IP}}$, and thus we have a 2-approximation algorithm.

4 Primal dual method

- Example: suppose min $4x_1 + 5x_2 + 7x_3$ subject to $x_1 + x_2 + x_3 \geq 2$ (1) and $x_1 + 2x_2 + 3x_3 \geq 8$ (2), $x_1, x_2, x_3 \geq 0$. Can I get a bound on this objective function?
- Multiply (1) by 4, and we get $4x_1 + 4x_2 + 4x_3 \geq 8$. The objective function is clearly bigger: $4x_1 + 5x_2 + 7x_3 \geq 8$.
- Multiply (2) by 2, and we get $4x_1 + 5x_2 + 7x_3 \geq 2x_1 + 4x_2 + 6x_3 \geq 16$.
- Take y_i multiples of equation number i, and then add them up. We get: $x_1(y_1 + y_2) + x_2(y_1 + 2y_2) + x_3(y_1 + 3y_2) \geq 2y_1 + 8y_2$. Comparing coefficients with the objective function, $y_1 + y_2 \leq 4$, $y_1 + 2y_2 \leq 5$, $y_1 + 3y_2 \leq 7$; maximise $2y_1 + 8y_2$; $y_1, y_2 \geq 0$.
- Dual problem: Maximise $b \cdot y$, subject to $A^T \cdot y \leq c^T$.
- The optimal value of the primal problem is the optimal value of the dual problem.
- Notice the relationship between the primal and the dual (see diagram).
- The dual problem of the vertex cover problem: maximise $\displaystyle\sum_{(i,j) \in E} y_{ij}$, subject to $\displaystyle\sum_{j:(i,j) \in E} y_{ij} \leq w_i \,\forall\, i \in V, y_{ij} \geq 0$
- The node is tight if the constraints become equalities, and tight nodes turn out to be the vertex cover.