COMP3610 Lecture Week 11

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1 Weighted interval scheduling with dynamic programming

1.1 Problem description

- Given requests $\{1, \ldots, n\}$, each request has start time s_i, finish time f_i and value v_i.
- Find a set $S \subseteq \{1, \ldots, n\}$ of compatible intervals that maximises $\sum_{i \in S} v_i$.

1.2 Analysis

- Order: $f_1 \leq f_2 \leq \ldots \leq f_n$, that is $i \leq j \Leftrightarrow f_i \leq f_j$.
- $p(j)$ is the largest index $i < j$ such that i, j are disjoint.
- Let O be the optimal solution.
- If $n \in O$, then $\{p(n) + 1, \ldots, n - 1\} \nsubseteq O$. We can then split the problem up into a subproblem. The optimal solution of a subproblem forms part of the solution to the problem: WIS$(\{1, \ldots, p(n)\}) \subseteq O$.
- If $n \notin O$, then $O =$ WIS$(\{1, \ldots, n - 1\})$.
- $O_j =$ WIS$(\{1, \ldots, j\})$ and OPT$(j) = \sum_{i \in O_j} v_i$
- OPT$(j) = \max\big(v_j + \text{OPT}(p(j)), \text{OPT}(j-1)\big)$: there are only two choices, that is, whether j is in the optimal solution or not. In other words, $j \in O_j$ iff $v_j + \text{OPT}(p(j)) \geq \text{OPT}(j - 1)$.
- In the recursion tree, there is wastage in repeating the same function calls. This can be replaced with memoization or an iterative technique. Assuming that the input has been sorted by finishing time, the result can be computed in $O(n)$ time.
- We also need to calculate the solution S in addition to its value. We use the fact that $j \in O_j$ iff $v_j + \text{OPT}(p(j)) \geq \text{OPT}(j - 1)$ to extract S out of the algorithm by reconstructing the choices in the algorithm; there is no need to continually recalculate S.

- There are only a polynomial number of subproblems.
- The solution to the original problem can be easily computed from the solutions to the subproblems. (For example, the original problem may actually be one of the subproblems.)
- There is a natural ordering on subproblems from ”smallest” to ”largest”, together with an easy-to-compute recurrence that allows one to determine the solution to a subproblem from the solution to some number of smaller subproblems.