COMP3610 Lecture Week 9

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1 Recapping

- Approximation algorithms:
  1. Run in polynomial time
  2. The solution (objective value) has to be provably "close" to the optimal
- Greedy approach used for load balancing and set cover.
- $T^* \leq T \leq kT^*$
- Start by deriving some lower bound for T^*, which must hold for any algorithm that we use.

2 Pricing method

- Vertex cover problem:
  - Given $G(V, E)$, a **vertex cover** is a set $S \subset V$ such that each edge is incident on at least one element of S.
  - A weight w_i is associated with each vertex $v_i \in V$.
  - The objective is to minimise the weight of the vertex cover:

$$W(S) = \sum_{i \in S} w_i$$

  - There is a mapping from vertex cover to the set cover problem. The edges become the set U to be covered, all the edges incident on each vertex become a subset S_i of U, and the weights of the vertices become the weights of S_i.
  - Previously, the approximation algorithm for set cover had a constant factor of $H(d^*)$, where d^* was the maximum size of the sets. We now present a 2-approximation algorithm.
- Approximation algorithm for the vertex cover problem:

– We want to associate a "price" with every edge e, $p_e \geq 0$.
Definition: A price p_e is **fair** if for each vertex i:

$$\sum_{e=(i,j)} p_e \leq w_i$$

– Claim: For any vertex cover S^* and any non-negative and fair price p_e:

$$\sum_{e \in E} p_e \leq w(S^*)$$

Proof:

$$\sum_{e=(i,j)} p_e \leq w_i \, \forall i \in S^*$$

$$\sum_{e} p_e \leq \sum_{i \in S^*} \sum_{e=(i,j)} p_e \leq \sum_{i \in S^*} w_i = w(S^*)$$

– Definition: A vertex i is **tight** if:

$$\sum_{e=(i,j)} p_e = w_i$$

– VERTEX-COVER-APPROX(G, w):
  1. Set $p_e = 0 \, \forall e \in E$
  2. While there is an $e = (i, j)$ such that neither i or j is tight:
     (a) Select such an e.
     (b) Increase p_e without violating fairness.
  3. Let S be the set of all tight vertices.
  4. Return S.

– Claim: The set S and prices returned by the algorithm satisfy:

$$w(S) \leq 2 \sum_{e \in E} p_e$$

Proof: All nodes in S are tight.

$$\sum_{e=(i,j)} p_e = w_i \, \forall i \in S$$

$$w(S) = \sum_{i \in S} w_i = \sum_{i \in S} \sum_{e=(i,j)} p_e \leq 2 \sum_{e \in E} p_e \leq 2w(S^*)$$

Thus, we have a 2-approximation algorithm.

– If you have a greedy algorithm that choose vertices in order of degree, you can have an approximation that is arbitrarily large.