MATH1901 Differential Calculus (Advanced)

Chapter 3: Functions

Definitions

$f : A \to B$

- A and B are sets
- *f* assigns to each element *x* in A *exactly one* element in B
- A is the *domain* of the function
- B is the *codomain* of the function
- The *range* is the set of all possible values of *f(x)* as *x* varies over the domain. It is a subset of the domain
- A vertical line should not cut the graph of a function more than once

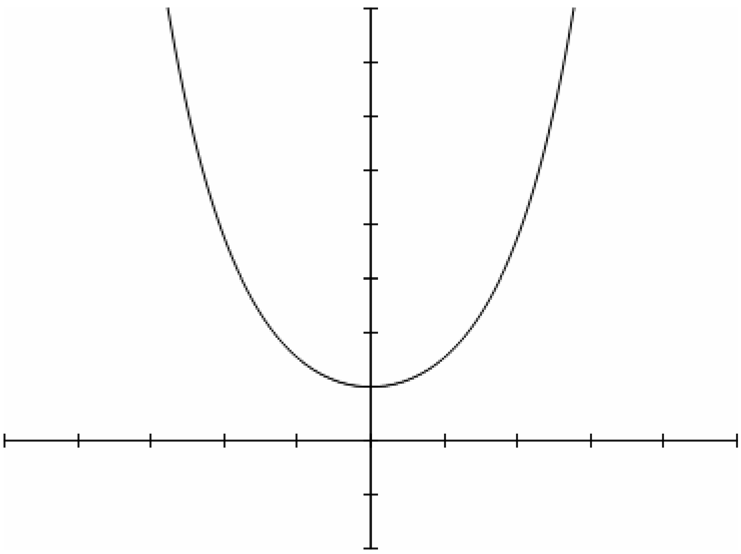
Composite Functions

$$(g \circ f)(x) = g(f(x))$$

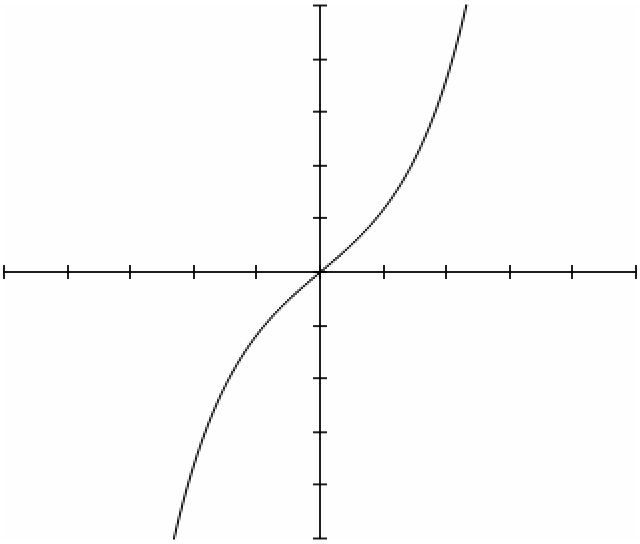
for all *x* in the domain of *f* such that *f(x)* is in the domain of g.

Hyperbolic Functions

$$\cosh x = \frac{e^x + e^{-x}}{2}$$



$$\sinh x = \frac{e^x - e^{-x}}{2}$$



Identities

- $\cosh^2 x - \sinh^2 x = 1$
- $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$
- $\cosh(x+y) = \sinh x \sinh y + \cosh x \cosh y$

Derivatives

- $\frac{d}{dx}\cosh x = \sinh$
- $\frac{d}{dx}\sinh x = \cosh x$

Injective Functions

- A function is *injective* (or *one-to-one*) if distinct elements in its domain are mapped to distinct elements in its codomain.

Tests

- $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ or $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
- The horizontal line test works if the function is from R to R.

Surjective Functions

- A function is *surjective* (or *onto*) if its range is equal to its codomain.

Bijective Functions

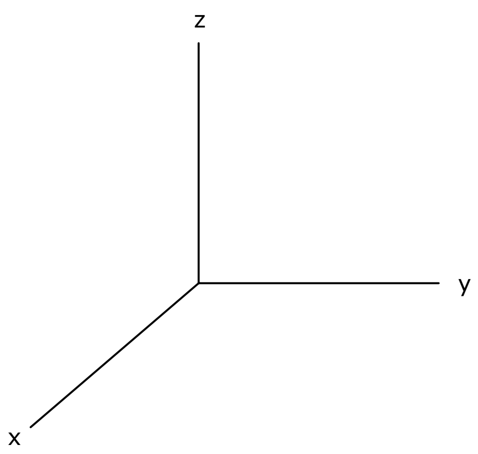
- A function is *bijective* if it is both injective and surjective.

Inverse Functions

- Two functions $f : A \to B$ and $f^{-1} : B \to A$ are *inverse functions* if $(f^{-1} \circ f)(x) = x$ for all $x \in A$ and $(f^{-1} \circ f)(x) = x$ for all $x \in B$
- A function has an inverse iff it is *bijective*.

Chapter 4: Curves and Surfaces in 3-dimensional Space

Cartesian Coordinates in 3 Dimensions



Curves in 3-Dimensional Space

- A *single equation* that relates *x*, *y* and *z* will, in general, represents a surface.
- *More than one equation* is needed to represent a curve in three dimensions. This is done by parametric equations.

Planes

- The Cartesian equation of a plane in 3-dimensional space is of the form *ax* + *by* + *c* = d, where not all of *a*, *b* and *c* are zero.

Functions of Two Real Variables

- The *domain* is a region in R^2 and the *codomain* is R.
- The *range* is the set of all possible values taken by *f(x, y)* as *(x, y)* varies over domain.

Chapter 5: Limits and the Limit Laws

Definition of a Limit

Intuitive Definition

- Suppose that *l* is a real number. The limit of *f(x)* as *x* approaches *c* is equal to *l*, or $\lim_{x \to c} f(x) = l$, if we can make *f(x)* as close as we like to *l* for all *x* sufficiently close to *c* (but not equal to *c*).

Mathematically Precise Definition

- Suppose that *l* is a real number. Then the limit of *f(x)*, as *x* approaches *c*, is equal to *l* if for any number $\varepsilon > 0$ we can always find a number $\delta > 0$ such that $|f(x) - l| < \varepsilon$ whenever $0 < |x - c| < \delta$.
- We first *choose* ε and then *find* a δ so that $|f(x) - l| < \varepsilon$ whenever $0 < |x - c| < \delta$. Then that limit holds true if for every possible $\varepsilon > 0$, we can always find such a δ.

The Squeeze Law

- A limit law: Suppose that $f(x) \le g(x)$, for all *x* near *c*. Then $\lim_{x \to c} f(x) \le \lim_{x \to c} g(x)$.
- The squeeze law: Suppose that $f(x) \le g(x) \le h(x)$, for all *x* near *c*, and that $\lim_{x \to c} f(x) = l$ and $\lim_{x \to c} h(x) = l$. Then $\lim_{x \to c} g(x) = l$.

Limits of Functions of Two Variables

Intuitive Definition

- $\lim_{(x,y) \to (c,d)} f(x) = l$ if we can make *f(x, y)* as close as we like to *l* for all *(x, y)* sufficiently close to, but not equal to, *(c, d)*.

Mathematically Precise Definition

- Suppose that *l* is a real number. Then, the limit of *f(x, y)*, as *(x, y)* approaches *(c, d)*, is equal to *l* if for each $\varepsilon > 0$ we can find a number $\delta > 0$ such that $|f(x, y) - l| < \varepsilon$ whenever $0 < \sqrt{(x - c)^2 + (y - d)^2} < \delta$.
- The limit should exist and be the same no matter which direction you come from.

Limits at Infinity

- Suppose that *l* is a real number. Then, the limit of *f(x)*, as *x* approaches ∞, is equal to *l* if for all $\varepsilon > 0$ there exists an $N > 0$ such that $|f(x) - l| < \varepsilon$ whenever $x > N$.
- The limit of *f(x)*, as *x* approaches −∞, is equal to *l* if for all $\varepsilon > 0$ there exists an $N > 0$ such that $|f(x) - l| < \varepsilon$ whenever $x < -N$.

Infinite Limits

- The limit of $f(x)$, as x approaches c, is equal to ∞ if for every positive number $N > 0$, there exists a $\delta > 0$ such that $f(x) > N$ whenever $0 < |x - c| < \delta$.
- The limit of $f(x)$, as x approaches c, is equal to $-\infty$ if for every positive number $N > 0$, there exists a $\delta > 0$ such that $f(x) < -N$ whenever $0 < |x - c| < \delta$.
- The limit laws do *not* apply to infinite limits.

Chapter 6: Continuous Functions

Introduction

- Continuous functions do not make sudden jumps or changes.
- If f is a continuous function and if c belongs to the domain of f then $\lim_{x \to c} f(x) = f(c)$.

Continuity at a Point

- Suppose that $f(x)$ is a function and that c is a point in the domain of $f(x)$. Then $f(x)$ is continuous at $x = c$ if the limit $\lim_{x \to c} f(x)$ and is equal to $f(c)$.
- If $f(x)$ is continuous at $x = c$, then $f(c)$ is finite.

Ways in which a function can fail to be continuous at a point

- The function has a hole at the point
- It jumps
- There is an asymptote

Continuous Functions

- A function is *continuous* if it is continuous at every point in its domain. That is, $f(x)$ is continuous if $\lim_{x \to c} f(x) = f(c)$ for all c in the domain of f.
- A function is *continuous everywhere* if it is continuous for all real c.

Composition Law

- Suppose that $f(x)$ is a continuous function and that $\lim_{x \to c} g(x) = l$. Then

  $$\lim_{x \to c} (f \circ g)(x) = f\left(\lim_{x \to c} g(x)\right) = f(l).$$

L'Hopital's Rule

- Suppose that $\lim_{x\to c} f(x)=0$, $\lim_{x\to c} g(x)=0$ and that the limit $\lim_{x\to c}\frac{f'(x)}{g'(x)}$ exists and is finite. Then $\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$.
- Suppose that $\lim_{x\to c} f(x)=\pm\infty$, $\lim_{x\to c} g(x)=\pm\infty$ and that the limit $\lim_{x\to c}\frac{f'(x)}{g'(x)}$ exists and is finite. Then $\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$.

The Intermediate Value Theorem

- Suppose that *f(x)* is continuous on the closed interval [*a*, *b*]. Then *f(x)* takes every value between *f(a)* and *f(b)* as *x* varies between *a* and *b*.
- Alternatively, if *k* is any number in between *f(a)* and *f(b)* then *f(x)* = *k* has at least one solution in the interval [*a*, *b*].
- The IVT can be applied to approximate the solution of an equation. Suppose that *f* is continuous on the interval [*a*, *b*] and that *f(a)* and *f(b)* have different signs. Then *f(c)* = 0, for some $c \in [a, b]$.

Chapter 7: Differentiable Functions

The Derivative of a Function

- A function *f* is differentiable at *x* = *c* if $f'(c)=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$ exists.
- The derivative of a function *f* is $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

Differentiable Functions and Continuity

- Every differentiable function is continuous.

Functions can fail to be differentiable if

- The function is not differentiable at a point in the given interval, and has a sharp point or a cusp.
- The function changes too rapidly near a particular point.

Derivative of an Inverse Function

- Suppose that *f* is a differentiable function. Then f^{1} is differentiable and

$$\frac{df^{-1}}{dx}=\frac{1}{f'\left(f^{-1}(x)\right)}.$$

The Extreme Value Theorem

- Suppose that *f* is continuous on the closed interval [*a*, *b*]. Then *f* has both a minimum and a maximum value on [*a*, *b*]; that is, there exist numbers *m* and *M* such that $m \le f(x) \le M$, for all $x \in [a, b]$, $m = f(c)$ and $M = f(d)$ for some $c, d \in [a, b]$.
- The *range* of *f* on [*a*, *b*] is [*m*, *M*].
- A function *f* has a *local minimum* at $x = c$ if there exists a $\delta > 0$ such that $f(x) \ge f(c)$ for all $x \in (c - \delta, c + \delta)$. The same goes for *local maximum*.
- Suppose that *f* is differentiable on [*a*, *b*] and that $c \in [a, b]$ is a maximum or a minimum value for *f* on [*a*, *b*]. Then either $c = a$, $c = b$ or $f'(c) = 0$.

The Mean Value Theorem

- Suppose that *f* is differentiable on [*a*, *b*]. Then there exists a number $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Chapter 8: Taylor Polynomials

Constructing Taylor Polynomials about x = 0

- The *n*th degree Taylor polynomial of a function *f* around *x* = 0 is

$$T_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n .$$

- The *n*th degree Taylor polynomial of a function *f* around *x* = *a* is

$$T_n(x) = f(a) + f'(a)x + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n .$$

The Remainder Term

- The difference between *f* and T_n is $R_n(x) = f(x) - T_n(x)$.
- The Lagrange form of the remainder is $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$.

Chapter 10: Partial Derivatives and Tangent Planes

- Let *f* be a function of two variables. Then the partial derivative of *f* with respect to *x* is

$$\frac{\partial f}{\partial x} = f_x(x, y) = \lim_{h \to 0}\left(\frac{f(x+h, y) - f(x, y)}{h}\right).$$

- Let *f* be a function of two variables. Then the partial derivative of *f* with respect to *y* is

$$\frac{\partial f}{\partial y} = f_y(x, y) = \lim_{h \to 0}\left(\frac{f(x, y+h) - f(x, y)}{h}\right).$$

- If (a, b) belongs to the domain of f, f_x and f_y exist and are continuous when $x = a$ and $y = b$ then f is differentiable at (a, b).

Higher Order Partial Derivatives

- $(f_x(x,y))_x = f_{xx}(x,y) = \dfrac{\partial^2 f}{\partial x^2}$
- $(f_x(x,y))_y = f_{xy}(x,y) = \dfrac{\partial^2 f}{\partial y \partial x}$
- $(f_y(x,y))_x = f_{yx}(x,y) = \dfrac{\partial^2 f}{\partial x \partial y}$
- $(f_y(x,y))_y = f_{yy}(x,y) = \dfrac{\partial^2 f}{\partial y^2}$

Tangent Planes

- The equation of the tangent plane is $z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$.

Linear Approximation Using Differentials

- The differential of a function of two variables is given by $dz = \dfrac{\partial z}{\partial x} dx + \dfrac{\partial z}{\partial y} dx$.

The Total Derivative Rule

- Suppose that $z = f(x, y)$ where $x = g(t)$ and $y = h(t)$ and f, g and h are all differentiable functions. Then the total derivative of z is given by $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$.

Implicit Differentiation

- $\dfrac{dy}{dx} = \dfrac{-f_x(x,y)}{f_y(x,y)}$

Chapter 12: Directional Derivatives and the Gradient Vector

The Gradient Vector and the Directional Derivative

- The gradient of f is $\text{grad } f(x,y) = \nabla f(x,y) = \dfrac{\partial f}{\partial x}\mathbf{i} + \dfrac{\partial f}{\partial y}\mathbf{j}$.
- The directional derivative of f in the direction of a unit vector u is

$$\begin{aligned} D_u f(x,y) &= \mathbf{u} \cdot \nabla f(x,y) = u_1 f_x(x,y) + u_2 f_y(x,y) \\ &= \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(x,y)}{h} \end{aligned}$$

Interpreting the Gradient Vector

- $D_u f(a,b) = |\nabla f(a,b)| \cos\theta$ where θ is the angle between ∇f and $\mathbf{u}$
- Hence the greatest value of the directional derivative is $|\nabla f(a,b)|$. The direction of steepest increase of *f(x, y)* is ∇f .
- It will take its largest negative value when *u* is in the opposite direction to ∇f .
- The tangent with zero slope lies in the direction at right angles to the gradient vector.
- ∇f is normal to the level curve *f(x, y)* = *c*.