MATH1903 Cheat Sheet

Upper and Lower Riemann Sums

- To find a definite integral using upper or lower Riemann sums, make the width of the divisions smaller and smaller.
- Telescoping sum method:

$$\begin{aligned} n^3 &= \left(n^3 - (n-1)^3\right) + \left((n-1)^3 - (n-2)^3\right) + \ldots + (2^3 - 1^3) + (1^3 - 0^3) \\ &= \sum_{i=1}^{n} \left(i^3 - (i-1)^3\right) \\ &= \sum_{i=1}^{n} i^3 - (i^3 - 3i^2 + 3i - 1) \\ &= \ldots \end{aligned}$$

Fundamental Theorem of Calculus

$$\frac{d}{dx} \int_a^x f(t)dt = f(x)$$

for any constant a

Trigonometric Substitutions

- Replace $a^2 - x^2$ with $x = a \sin \theta$
- Replace $a^2 + x^2$ with $x = a \tan \theta$
- Replace $x^2 - a^2$ with $x = a \sec \theta$

Volumes of Revolution

- Disk method:

$$V = \int_a^b \pi (f(x))^2 fx$$

- Shell method:

$$V = 2\pi \int_a^b x f(x) dx$$

Length of an Arc

- Parametric form:

$$l = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

- Cartesian form:

$$l = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Monotone Convergence Theorem

- An infinite non-decreasing sequence of real numbers bounded about has a well-defined limit.

Logarithms and Exponentials

- Define:
  $a^b = e^{b \ln a}$
- $\ln(a^n) = n \ln(a)$
- $\ln x = \int_1^x \frac{1}{t} dt$

Series

- Sum of geometric series: $\sum ar^k = \frac{a}{1-r}$ for |r| < 1 (if |r| ≥ 1, series diverges)
- Harmonic series diverges
- Ratio test: $L = \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right|$, converges if L < 1, diverges if L > 1
- Power series may be differentiated like ordinary polynomials

Maclaurin Series

- $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$
- Memorise:
  - $e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$
  - $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$
  - $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$
  - $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots$
  - $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$

Taylor Series

- $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$
- $\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \ldots$
- Proof of above: $\frac{1}{x} = \frac{1}{1-(1-x)}$ = geometric series → anti-differentiate

Logistic Equation

- $\frac{dy}{dt} = ay(b-y)$
- $y = \frac{b}{1+Ke^{-bat}}$
- Equilibrium solutions: y = 0, y = b
- Population grows most rapidly at when population is b/2
- Equilibrium solutions given by $\frac{dy}{dt} = 0$ → check sign around it for stable/unstable

First Order Linear Differential Equations

- $F(x) = e^{\int p(x)dx}$
- $yF(x) = \int F(x)r(x)dx$

Second Order Linear Differential Equations

- Complementary functions:
  - $y = C_1e^{\lambda_1 x} + C_2e^{\lambda_{21} x}$ or
  - $y = C_1e^{\lambda_1 x} + C_2xe^{\lambda_{21} x}$ or
  - $y = C_1e^{ax}\sin bx + C_2e^{ax}\cos bx$
- Trial functions:
  - Constant: y = A
  - Linear: y = Ax + B
  - Quadratic: y = Ax^2 + Bx + c
  - $k_1e^{k_2x} : y = Ae^{k_2x}$
  - Sines and cosines: $y = A\sin ax + B\cos ax$
- Multiply by x if trial function overlaps with complementary function
- General solution: y = (complementary function) + (particular solution)

Solving Simultaneous Equations

- Find y in terms of x and $\frac{dx}{dt}$
- $\frac{dy}{dt}$ in terms of x and $\frac{dx}{dt}$
- Differentiate to get $\frac{d^2x}{dt^2}$ in terms of $\frac{dx}{dt}$ and $\frac{dy}{dt}$
- $\frac{d^2x}{dt^2}$ in terms of x and $\frac{dx}{dt}$