Question 1

(a)

Firstly, note that

$$\begin{aligned} (n+1) - \sqrt{n(n+1)} &= (n+1) - \sqrt{n^2+n} \\ &> (n+1) - \sqrt{n^2+n+\frac{1}{4}} \\ &= (n+1) - (n+\frac{1}{2}) \\ &= \frac{1}{2} \end{aligned}$$

Hence,

$$\sqrt{n+1} - \sqrt{n} > \frac{1}{2\sqrt{n+1}} \qquad (1)$$

Secondly, note that

$$\begin{aligned} \sqrt{n(n+1)} - n &= \sqrt{n^2+n} - n \\ &< \sqrt{n^2+n+\frac{1}{4}} - n \\ &= (n+\frac{1}{2}) - n \\ &= \frac{1}{2} \end{aligned}$$

Hence,

$$\sqrt{n+1} - \sqrt{n} < \frac{1}{2\sqrt{n}} \qquad (2)$$

Combining (1) and (2), we obtain

$$\frac{1}{2\sqrt{n+1}} < \sqrt{n+1} - \sqrt{n} < \frac{1}{2\sqrt{n}} \qquad (3)$$

(b)

By substituting $k = n+1$ in the left inequality of (3), we obtain

$$\begin{aligned}
\frac{1}{2\sqrt{k}} &< \sqrt{k}-\sqrt{k-1} \\
\frac{1}{\sqrt{k}} &< 2\sqrt{k}-2\sqrt{k-1}
\end{aligned} \tag{4}$$

Both sides of the inequality are positive, and so, for $n \in \mathbb{N}$,

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{\sqrt{k}} &< \sum_{k=1}^{n} \left(2\sqrt{k}-2\sqrt{k-1}\right) \\
s_n &< 2\sum_{k=1}^{n} \left(\sqrt{k}-\sqrt{k-1}\right) \\
&= 2\left[\left(\sqrt{1}-\sqrt{0}\right)+\left(\sqrt{2}-\sqrt{1}\right)+\cdots+\left(\sqrt{n}-\sqrt{n-1}\right)\right] \\
&= 2\sqrt{n} \\
&= x_n
\end{aligned} \tag{5}$$

Also, by substituting $k = n+1$ in the right inequality of (3), we obtain

$$\begin{aligned}
\sqrt{k}-\sqrt{k-1} &< \frac{1}{2\sqrt{k-1}} \\
2\sqrt{k}-2\sqrt{k-1} &< \frac{1}{\sqrt{k-1}}
\end{aligned}$$

Both sides of the inequality are positive, and so, for $n \in \mathbb{N}$,

$$\begin{aligned}
2\sum_{k=2}^{n} \left(\sqrt{k}-\sqrt{k-1}\right) &< \sum_{k=2}^{n} \frac{1}{\sqrt{k-1}} \\
2\left[\left(\sqrt{2}-\sqrt{1}\right)+\left(\sqrt{3}-\sqrt{2}\right)+\cdots+\left(\sqrt{n}-\sqrt{n-1}\right)\right] &< \sum_{k=1}^{n-1} \frac{1}{\sqrt{k}} \\
2\left(\sqrt{n}-1\right) &< \sum_{k=1}^{n} \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{n}} \\
x_n - 2 &< s_n - \frac{1}{\sqrt{n}} \\
x_n - 2 + \frac{1}{\sqrt{n}} &< s_n
\end{aligned} \tag{6}$$

When we combine (5) and (6) and divide everything by x_n, we get

$$1 - \frac{2}{x_n} + \frac{1}{\sqrt{n}x_n} < \frac{s_n}{x_n} < 1$$

Hence, by the Squeeze Law,

$$\lim_{n\to\infty} \frac{s_n}{x_n} = 1 \qquad (7)$$

and so $s_n \sim x_n$ as required.

Question 2

(a)

(⇒)
We first show that $f_A(x) = 0 \Rightarrow x \in \overline{A}$.
By assumption, $\inf_{z \in A} \|x - z\| = 0$.
If $x \in \overline{A}$, then for every $r > 0$, $B(x, r) \cap A \neq \varnothing$. For this, we give a proof by contradiction.
Suppose that there are no elements of A in $B(x, \epsilon)$, for some $\epsilon > 0$. If this were the case, as f_A describes the greatest lower bound of the 'distance' between x and any element of A, $f_A(x) = \inf_{z \in A} \|x - z\| = \epsilon$, which is not the case.
So therefore, $x \in \overline{A}$ as required.
(⇐)
We next show that $f_A(x) = 0 \Leftarrow x \in \overline{A}$.
Since $A \in \overline{A}$, $B(x, r) \cap A \neq \varnothing$ for all $r > 0$. We can take r to be arbitrarily small; for example, we can take a sequence of balls $B_n(x, \frac{1}{n})$. Clearly, $\inf_{z \in A} \|x - z\| = 0$.

(b)

The hint can be rearranged as

$$\|x - z\| - \|y - z\| \quad \leq \quad \|x - y\|$$

for all $x, y, z \in \mathbb{K}^n$.
Hence,

$$\begin{aligned} |f_A(x) - f_A(y)| &= |\inf_{z \in A} \|x - z\| - \inf_{z \in A} \|y - z\| \,| \\ &= |\inf_{z \in A} (\|x - z\| - \|y - z\|) \,| \\ &\leq |\inf_{z \in A} \|x - y\| \,| \\ &\leq \|x - y\| \end{aligned} \tag{8}$$

If $f_A \in C(\mathbb{K}^N, \mathbb{R})$, $\lim_{y \to x} f_A(y) = f_A(x)$, or equivalently $|f_A(x) - f_A(y)| \to 0$ as $y \to x$. Now, as $x \to y$, $\|x - y\| \to 0$. By applying the Squeeze Law to (8), $|f_A(x) - f_A(y)| \to 0$ as required.

(c)

Since f is to be continuous and change value as you travel 'away' from A and 'towards' B (we are guaranteed that there is a 'gap' between the sets because they are non-empty and closed, and $A \cap B = \varnothing$), it bears resemblance to the division of a line into certain proportions (see the following diagram). It thus motivates a formula for f:

$$f(x) = \frac{f_A(x)}{f_A(x) + f_B(x)}$$

This function f has the following properties:

- f is continuous because it involves the sum and quotient of continuous functions. Note that f_A and f_B are never simultaneously 0, so it is not possible for the denominator to be 0.
- When $x \in A$, $f = \frac{0}{0+f_B(x)} = 0$ as required.
- When $x \in B$, $f = \frac{f_A(x)}{f_A(x)+0} = 1$ as required.

Question 3

(a)

In the following discussion, we may assume that $x \in D$ since D is closed and bounded, and so all convergent sequences in D converge to a point in D.
($\Rightarrow$)
We will first show that $f_n(x_n) \to f(x)$ for every sequence in D with $x_n \to x \Rightarrow f_n \to f$ uniformly on D.
Fix $\epsilon > 0$.
If there exists $n_0 \in \mathbb{N}$ such that

$$\|f_n(x) - f(x)\| < \epsilon$$

for all $n > n_0$ and $x \in D$, then $f_n \to f$ uniformly.
Now, choose an arbitrary sequence in D, where $x_n \to x$. By assumption, $\|f_n(x_n) - f(x)\| < \frac{\epsilon}{2}$ whenever $n > n_0^*$, where n_0^* is defined as follows:

$$n_0^* = \sup_{x_n \text{in} D} \{n_0 \mid \|f_n(x_n) - f(x)\| < \epsilon \text{ for } n > n_0\}$$

Note that n_0^* must exist since $f_n(x_n) \to f(x)$ converges for every sequence in D with $x_n \to x$ by assumption.
Also, since $f_n \in C(D, \mathbb{R}^N)$, $x_n \to x \Rightarrow f_n(x_n) \to f_n(x)$. That is to say, there exists $\delta > 0$ and $n_1 \in \mathbb{N}$ such that $\|f_n(x_n) - f_n(x)\| < \frac{\epsilon}{2}$ whenever $\|x_n - x\| < \delta$, which occurs whenever $n > n_1$.
Hence,

$$\begin{aligned} \|f_n(x) - f(x)\| &\leq \|f_n(x_n) - f_n(x)\| + \|f_n(x_n) - f(x)\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

for $n > \max\{n_0^*, n_1\}$.
Since $\epsilon > 0$ arbitrary, and because the argument above applies to the same n_0^* for all sequences x_n in D (and hence all x in D), $f_n \to f$ uniformly on D.
($\Leftarrow$)
We will next show that $f_n(x_n) \to f(x)$ for every sequence in D with $x_n \to x \Leftarrow f_n \to f$ uniformly on D.
Fix $\epsilon > 0$ arbitrary.
By assumption, $f_n \to f$ uniformly, so for every $\epsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that $\|f_n(x) - f(x)\| < \frac{\epsilon}{2}$ for all $n > n_1$, $x \in D$. Also, by assumption, $x_n \to x$, so from above, $\|f_n(x_n) - f_n(x)\| < \frac{\epsilon}{2}$ for $n > n_2$.
Hence,

$$\begin{aligned} \|f_n(x_n) - f(x)\| &\leq \|f_n(x_n) - f_n(x)\| + \|f_n(x) - f(x)\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

for all $n > \max\{n_1, n_2\}$.
Since $\epsilon > 0$ arbitrary, $f_n(x_n) \to f(x)$.

(b)

We shall prove that $f_n(x_n) \to f(x)$ for every sequence in D where $x_n \to x \Leftrightarrow f_n$ converges locally uniformly to f. Combining this result with the result in part (a) will prove that $f_n \to f$ uniformly on $D \Leftrightarrow f_n$ converges locally uniformly to f.
($\Leftarrow$)
Suppose f_n converges locally uniformly to f. We need to prove that, given a sequence $x_n \to x$ in D, $f_n(x_n) \to f(x)$.
Now, fix a sequence x_n (and hence fix a value of x). By assumption of locally uniform convergence, for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ and $r > 0$ such that

$$|f_n(y) - f(y)| < \frac{\epsilon}{2}$$

for $n > n_0$ and all $y \in B(x, r) \cap D$.
In particular,

$$\|f_n(x) - f(x)\| < \frac{\epsilon}{2}$$

for $n > n_0$ since $x \in B(x, r) \cap D$.
Also, from above, for every $\epsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that

$$\|f_n(x_n) - f_n(x)\| < \frac{\epsilon}{2}$$

for $n > n_1$.
Hence,

$$\begin{aligned} \|f_n(x_n) - f(x)\| &\leq \|f_n(x_n) - f_n(x)\| + \|f_n(x) - f(x)\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

for $n > \max\{n_0, n_1\}$.
That is, $f_n(x_n) \to f(x)$ for all sequences in D where $x_n \to x$.
($\Rightarrow$)
Suppose that $f_n(x_n) \to f(x)$ for every sequence in D with $x_n \to x$. We are to show that f_n converges locally uniformy to f.
Fix a sequence $x_n \to x$, and so we have fixed x also.
Note that $B(x, r) \cap D \neq \varnothing$ for all $r > 0$. In addition, because $x_n \to x$, there exists $n_0 in \mathbb{N}$ such that $x_n \in B(x, r) \cap D$ for $n > n_0$, so we can truncate the elements of x_n that do not lie within $B(x, r) \cap D$. The new sequence will lie entirely within $B(x, r) \cap D$.
Setting D to be $B(x, r) \cap D$ in the result of (a), $f_n \to f$ on $B(x, r) \cap D$ for all $r > 0$, and hence f_n converges locally uniformly to f.