MATH3964 Assignment 1

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Question 1

i)

Using the lemma from the lectures, which applies to a curve around a pole of order 1:

$$\lim_{r\to 0}\int_{S_r} f(z)\,dz = i(\beta-\alpha)\,\mathrm{Res}(f,a)$$

it is clear that if we set $(\beta-\alpha)=\pi$ and $\mathrm{Res}(f,a)=A$:

$$\lim_{r\to 0}\int_{S_r} f(z)\,dz = i\pi A$$

ii)

Let us investigate the complex-valued function:

$$f(z)=\frac{1}{z^4+1}$$

This has 4 simple poles at the solutions to $z^4=-1$: $z_1=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i$, $z_2=-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i$, $z_3=-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i$ and $z_4=\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i$.

Let S_R be $z=re^{i\theta}$, with $R>1$ (since $|z_i|=1$ for $i=1,2,3,4$) and $0\le\theta\le\pi$. Let S_L be the interval on the real axis $[-R,R]$. Let $S=S_L+S_R$.

Now, we can write:

$$f(z) = \frac{1}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}$$

and so:

$$\begin{aligned}
\text{Res}(f, z_1) &= \lim_{z\to z_1} (z-z_1)f(z) \\
&= \lim_{z\to z_1} \frac{1}{(z-z_2)(z-z_3)(z-z_4)} \\
&= \frac{1}{(\sqrt{2}i)(\sqrt{2})(\sqrt{2}+\sqrt{2}i)} \\
&= \frac{1}{-2\sqrt{2}+2\sqrt{2}i} \cdot \frac{-2\sqrt{2}-2\sqrt{2}i}{-2\sqrt{2}-2\sqrt{2}i} \\
&= \frac{-1-i}{4\sqrt{2}} \\
\text{Res}(f, z_2) &= \lim_{z\to z_2} (z-z_2)f(z) \\
&= \lim_{z\to z_2} \frac{1}{(z-z_1)(z-z_3)(z-z_4)} \\
&= \frac{1}{(-\sqrt{2})(-\sqrt{2}+\sqrt{2}i)(\sqrt{2}i)} \\
&= \frac{1}{2\sqrt{2}+2\sqrt{2}i} \cdot \frac{2\sqrt{2}-2\sqrt{2}i}{2\sqrt{2}-2\sqrt{2}i} \\
&= \frac{1-i}{4\sqrt{2}}
\end{aligned}$$

By the residue theorem,

$$\begin{aligned}
\oint_S \frac{1}{z^4+1}\,dz &= 2\pi i\left(\frac{-1-i}{4\sqrt{2}} + \frac{1-i}{4\sqrt{2}}\right) \\
&= 2\pi i \frac{-2i}{4\sqrt{2}} \\
&= \frac{\pi}{\sqrt{2}}
\end{aligned}$$

Let us bound $\int_{S_R} \frac{1}{z^4+1}\,dz$. Using $||z|-|w|| \le |z+w|$, $R^4 - 1 \le \left|R^4 e^{i4\theta} + 1\right|$ for $R > 1$. Therefore:

$$|f(z)| = \left|\frac{1}{R^4 e^{i4\theta}+1}\right| \le \frac{1}{R^4-1}$$

So, by the *ML* formula,

$$\left|\int_{S_R} \frac{1}{z^4+1}\,dz\right| \le \frac{1}{R^4-1} \cdot \pi R \to 0$$

as $R \to \infty$, and thus:

$$\lim_{R\to\infty} \int_{S_R} f(z)\,dz = 0$$

Hence:

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{x^4+1}\, dz &= \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{x^4+1}\, dz \\
&= \lim_{R\to\infty} \int_{S_L} f(z)\, dz \\
&= \lim_{R\to\infty} \int_{S_L} f(z)\, dz + \lim_{R\to\infty} \int_{S_R} f(z)\, dz \\
&= \lim_{R\to\infty} \oint_{S} f(z)\, dz \\
&= \frac{\pi}{\sqrt{2}}
\end{aligned}$$

Question 2

i)

Let us construct the power series for $\cos(\sqrt{z})$ from the power series for $\cos z$:

$$\begin{aligned}
\cos z &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \\
\cos \sqrt{z} &= 1 - \frac{z}{2!} + \frac{z^2}{4!} - \frac{z^3}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{n}}{(2n)!}
\end{aligned}$$

We shall show that we have absolute convergence for the non-negative real series:

$$\sum_{n=0}^{\infty} \frac{|z^n|}{(2n)!}$$

The series obviously gives $\cos(\sqrt{0}) = 1$ when $z = 0$. When $z \neq 0$, write $u_n = \frac{|z^n|}{(2n)!}$, and the ratio test gives:

$$\begin{aligned}
\lim_{n\to\infty} \frac{u_{n+1}}{u_n} &= \lim_{n\to\infty} \left(\frac{|z^{n+1}|}{(2n+2)!} \cdot \frac{(2n)!}{|z^n|} \right) \\
&= \lim_{n\to\infty} \frac{|z|}{(2n+1)(2n+2)} \\
&= 0 < 1
\end{aligned}$$

$\forall z \in \mathbb{C}\backslash\{0\}$. Hence, the power series for $\cos(\sqrt{z})$ converges absolutely for every $z \in \mathbb{C}$. Therefore the radius of convergence is ∞, and it is an entire function.
Now, the $(n+1)$-th term in the Taylor series is:

$$\frac{f^{(n)}(0)}{n!} z^n$$

Noting that the power series for an analytic function is the Taylor series due to uniqueness, and that the coefficient of z^n in the power series is $\frac{(-1)^n}{(2n)!}$, we have:

$$\begin{aligned} f^{(n)}(0) &= \frac{(-1)^n n!}{(2n)!} \\ &= \frac{(-1)^n}{(n+1)(n+2)\dots(2n)} \end{aligned}$$

ii)

The singularities of $\frac{1}{2\cos z - 1}$ are given by the solutions to $\cos(z) = \frac{1}{2}$, which are $\frac{\pi}{3} \pm 2k\pi$ and $\frac{-\pi}{3} \pm 2k\pi$ for $k \in \mathbb{Z}$. The radius of convergence is the distance from the centre to the nearest singularity. If the centre is $z = 0$, the nearest singularities are found at $\frac{\pi}{3}$ and $\frac{-\pi}{3}$. Therefore, the radius of convergence is $\frac{\pi}{3}$.

Question 3

i)

Take Γ to be a closed path. By the argument principle:

$$\oint_\Gamma \frac{f'(z)}{f(z)}\,dz = 2\pi i N$$

where N is the number of zeroes of $f(z)$, noting that there are no poles inside Γ since f is given to be analytic on U.
Now:

$$\begin{aligned} \oint_\Gamma \frac{f'(z)}{f(z)}\,dz &= \int_0^1 \frac{f'(\gamma(t))}{f(\gamma(t))}\gamma'(t)\,dt \\ &= \int_0^1 \frac{f'(\gamma(t))}{f(\gamma(t))}\,d(\gamma(t)) \\ &= \Big[\log(f(\gamma(t)))\Big]_0^1 \\ &= \log(f(\gamma(1))) - \log(f(\gamma(0))) \\ &= (\log|f(\gamma(1))| + i\arg(f(\gamma(1)))) - (\log|f(\gamma(0))| + i\arg(f(\gamma(0)))) \end{aligned}$$

Since Γ is a closed curve, we end up at the same distance away from the origin, and so $\log|f(\gamma(1))| = \log|f(\gamma(0))|$. Then:

$$\begin{aligned} \oint_\Gamma \frac{f'(z)}{f(z)}\,dz &= i\arg(f(\gamma(1))) - i\arg(f(\gamma(0))) \\ &= i\Delta_\Gamma \arg(f) = 2\pi i N \\ N &= \frac{1}{2\pi}\Delta_\Gamma \arg(f) \end{aligned}$$

ii)

(ignored as per instructions)