MATH3964 Assignment 2

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Question 1

a)

Given the Riccati equation:

$$w' = Pw^2 + Qw + R$$

we can express it as a cubic polynomial in w:

$$\begin{aligned} w'' &= P \cdot 2ww' + P'w^2 + Qw' + Q'w + R' \\ &= 2Pw\big[Pw^2 + Qw + R\big] + P'w^2 + Q\big[Pw^2 + Qw + R\big] + Q'w + R' \\ &= \big[2P^2\big]w^3 + \big[2PQ + P' + PQ\big]w^2 + \big[2PR + Q^2 + Q'\big]w + \big[QR + R'\big] \end{aligned} \tag{1}$$

b)

We equate the expression for w'' above with the given differential equation:

$$w''' = \frac{1}{2w}(w')^2 + \frac{3w^2}{2} + 4zw^2 + 2(z^2 - \alpha)w - \frac{\beta^2}{2w} \tag{2}$$

Comparing the coefficients of w^3:

$$\begin{aligned} \frac{3}{2} + \frac{P^2}{2} &= 2P^2 \\ P &= \epsilon_1 \end{aligned} \tag{3}$$

where ϵ_1 denotes ± 1.

Comparing the coefficients of $\frac{1}{w}$:

$$\begin{aligned} \frac{R^2}{2w} - \frac{\beta^2}{2w} &= 0 \\ R &= \epsilon_2 \beta \end{aligned} \tag{4}$$

where ϵ_2 denotes ± 1.

Comparing the coefficients of w^2:

$$\begin{aligned} \frac{2PQ}{2} + 4z &= 3PQ + P' \\ 4z &= 2\epsilon_1 Q \\ Q &= \frac{2z}{\epsilon_1} \end{aligned} \tag{5}$$

Comparing the coefficients of w:

$$\begin{aligned}
\frac{2PR+Q^2}{2}+2(z^2-\alpha) &= 2PR+Q^2+Q' \\
2z^2-2\alpha &= PR+\frac{Q^2}{2}+Q' \\
&= \epsilon_1\epsilon_2\beta+\frac{2z^2}{\epsilon_1^2}+\frac{2}{\epsilon_1} \\
-2\alpha-\frac{2}{\epsilon_1} &= \epsilon_1\epsilon_2\beta \qquad \text{since } \epsilon_1^2=1 \\
\beta &= \frac{-2\alpha}{\epsilon_1\epsilon_2}-\frac{2}{\epsilon_1^2\epsilon_2} \\
&= \frac{-2\alpha}{\epsilon_1\epsilon_2}-\frac{2}{\epsilon_2} \qquad (6)
\end{aligned}$$

which provides a constraint on the parameters α and β.

c)

Let us transform the Riccati equation into a second-order linear differential equation. Let $s = wP$. We see that s satisfies the differential equation:

$$s' = s^2+\left(Q+\frac{P'}{P}\right)s+PR \qquad (7)$$

because:

$$\begin{aligned}
s' &= wP'+w'P \\
&= wP'+(Pw^2+Qw+R)P \\
&= P^2w^2+(PQ+P')w+PR \\
&= s^2+\left(Q+\frac{P'}{P}\right)s+PR
\end{aligned}$$

Now, let $s=\frac{-u'}{u}$:

$$\begin{aligned}
s' &= -\frac{uu''+(u')^2}{u^2} \\
&= -\frac{u''}{u}+\left(\frac{u'}{u}\right)^2 \\
&= -\frac{u''}{u}+s^2 \\
\frac{u''}{u} &= s^2-s' = -PR-\left(Q+\frac{P'}{P}\right)s \\
u''-\left(Q+\frac{P'}{P}\right)u'+PRu &= 0 \\
u''-\frac{2z}{\epsilon_1}u'+\epsilon_1\epsilon_2\beta u &= 0 \qquad (8)
\end{aligned}$$

The solutions to the original Riccati equation can be expressed in terms of the solutions of (8):

$$w = -\frac{u'}{Pu} = -\frac{u'}{\epsilon_1 u} \tag{9}$$

However, we would like to express the differential equation in the form:

$$\bar{u}'' + f(z)\bar{u} = 0 \tag{10}$$

To achieve this, we use a gauge transformation:

$$u = \lambda\bar{u} \tag{11}$$
$$u' = \lambda\bar{u}' + \lambda'\bar{u} \tag{12}$$
$$u'' = \lambda\bar{u}'' + 2\lambda'\bar{u}' + \lambda''\bar{u} \tag{13}$$

Substitute these into (8):

$$\begin{aligned}
(\lambda\bar{u}'' + 2\lambda'\bar{u}' + \lambda''\bar{u}) - 2\epsilon_1 z(\lambda\bar{u}' + \lambda'\bar{u}) + \epsilon_1\epsilon_2\beta(\lambda\bar{u}) &= 0 \\
\bar{u}'' + 2\frac{\lambda'}{\lambda}\bar{u}' + \frac{\lambda''}{\lambda}\bar{u} - 2\epsilon_1 z\bar{u}' - 2\epsilon_1 z\frac{\lambda'}{\lambda}\bar{u} + \epsilon_1\epsilon_2\beta\bar{u} &= 0 \\
\bar{u}'' + \bar{u}'\left(2\frac{\lambda'}{\lambda} - 2\epsilon_1 z\right) + \bar{u}\left(\frac{\lambda''}{\lambda} - 2\epsilon_1 z\frac{\lambda'}{\lambda} + \epsilon_1\epsilon_2\beta\right) &= 0 \qquad (14)
\end{aligned}$$

We compare the coefficients of (10) and (14). By comparing the coefficients of $\bar{u}'$:

$$\begin{aligned}
2\frac{\lambda'}{\lambda} - 2\epsilon_1 z &= 0 \\
\frac{\lambda'}{\lambda} &= \epsilon_1 z \qquad (15) \\
\lambda &= e^{\frac{\epsilon_1 z^2}{2}} \\
\lambda' &= \epsilon_1 z e^{\frac{\epsilon_1 z^2}{2}} \\
\lambda'' &= e^{\frac{\epsilon_1 z^2}{2}}(z^2 + \epsilon_1) \\
\frac{\lambda''}{\lambda} &= z^2 + \epsilon_1 \qquad (16) \\
& \qquad (17)
\end{aligned}$$

By comparing the coefficients of $\bar{u}$:

$$\begin{aligned}
f(z) &= \frac{\lambda''}{\lambda} - 2\epsilon_1 z\frac{\lambda'}{\lambda} + \epsilon_1\epsilon_2\beta \\
&= z^2 + \epsilon_1 - 2\epsilon_1 z(\epsilon_1 z) + \epsilon_1\epsilon_2\beta \qquad \text{using (15) and (16)} \\
&= -z^2 + \epsilon_1 + \epsilon_1\epsilon_2\beta \qquad (18)
\end{aligned}$$

In order to express the solutions of the Riccati equation in terms of the solutions of $\bar{u}'' + f(z)\bar{u} = 0$:

$$w = \frac{-u'}{\epsilon_1 u} = \frac{\lambda'\bar{u} - \lambda\bar{u}'}{\epsilon_1\lambda\bar{u}} \tag{19}$$

where $\bar{u}$ is a solution of $\bar{u}'' + f(z)\bar{u} = 0$.

Question 2

a)

We are given:

$$J_1 = w' - \frac{w^2}{6} \tag{20}$$
$$J_2 = \frac{dJ_1}{dz} - \frac{2}{3}wJ_1 \tag{21}$$
$$J_3 = \frac{dJ_2}{dz} - wJ_2 \tag{22}$$

Hence:

$$\begin{aligned} J_2 &= \left(w'' - \frac{ww'}{3}\right) - \frac{2}{3}w\left(w' - \frac{w^2}{6}\right) \\ &= w'' - ww' + \frac{w^3}{9} \end{aligned} \tag{23}$$

$$\begin{aligned} J_3 &= \left(w''' - \left(ww'' + (w')^2\right) + \frac{3w^2w'}{9}\right) - w\left(w'' - ww' + \frac{w^3}{9}\right) \\ &= w''' - ww'' - (w')^2 + \frac{w^2w'}{3} - ww'' + w^2w' - \frac{w^4}{9} \\ &= w''' - 2ww'' + \frac{4w^2w'}{3} - (w')^2 - \frac{w^4}{9} \end{aligned} \tag{24}$$

Therefore, if we substitute the results into the equation:

$$J_3 + 4(J_1)^2 = 0 \tag{25}$$

we will get:

$$\left(w''' - ww'' - 2ww'' + \frac{4w^2w'}{3} - (w')^2 - \frac{w^4}{9}\right) + 4\left((w')^2 - \frac{w^2w'}{3} + \frac{w^4}{36}\right) = 0$$
$$w''' - 2ww'' + 3(w')^2 = 0 \tag{26}$$

which is the Chazy-III equation.

b)

Because u and u_1 are linearly independent solutions of the differential equation $\ddot{u} + p(t)\dot{u} + q(t)u = 0$, we have:

$$\ddot{u} + p(t)\dot{u} + q(t)u = 0$$
$$\ddot{u_1} + p(t)\dot{u_1} + q(t)u_1 = 0$$

which become, respectively:

$$u_1\ddot{u} + p(t)u_1\dot{u} + q(t)u_1u = 0 \tag{27}$$
$$u\ddot{u_1} + p(t)u\dot{u_1} + q(t)uu_1 = 0 \tag{28}$$

Subtracting (28) − (27), we obtain:

$$\begin{aligned}(u\ddot{u}_1 - u_1\ddot{u}) + p(t)(u\dot{u}_1 - u_1\dot{u}) &= 0\\ u\ddot{u}_1 - u_1\ddot{u} &= -p(t)(u\dot{u}_1 - u_1\dot{u})\\ \dot{D} &= -p(t)D \qquad (29)\end{aligned}$$

where $D = u\dot{u}_1 - u_1\dot{u}$.
Let $D' = k\exp\left\{-\int p(t)\,dt\right\}$:

$$\dot{D}' = -p(t)k\exp\left\{-\int p(t)\,dt\right\} = -p(t)D' \qquad (30)$$

Equations (29) and (30) lead to the conclusion that:

$$D = D' = k\exp\left\{-\int p(t)\,dt\right\} \qquad (31)$$

c)

From the parametric equations:

$$z = \frac{u_1}{u}, \qquad w(z) = \frac{6u\dot{u}}{D}$$

we obtain:

$$\begin{aligned}\dot{z} &= \frac{u\dot{u}_1 - u_1\dot{u}}{u^2}\\ &= \frac{D}{u^2} \qquad (32)\\ \dot{w} &= \frac{D\left\{6\big(u\ddot{u} + (\dot{u})^2\big)\right\} - 6u\dot{u}\dot{D}}{D^2}\\ &= \frac{6}{D^2}\left(Du\ddot{u} + D(\dot{u})^2 - \dot{D}u\dot{u}\right) \qquad (33)\end{aligned}$$

Applying the chain rule, we obtain:

$$\begin{aligned}w' &= \frac{\dot{w}}{\dot{z}}\\ &= \frac{6u^2}{D^3}\left(Du\ddot{u} + D(\dot{u})^2 - \dot{D}u\dot{u}\right) \qquad (34)\end{aligned}$$

Hence, we find an expression for J_1:

$$\begin{aligned}J_1 &= w' - \frac{w^2}{6}\\ &= \frac{6u^2}{D^3}\left(Du\ddot{u} + D(\dot{u})^2 - \dot{D}u\dot{u}\right) - \frac{6u^2(\dot{u})^2D}{D^3}\\ &= \frac{6u^2}{D^3}\left(Du\ddot{u} - \dot{D}u\dot{u}\right)\\ &= \frac{6u^3}{D^3}\left(D\ddot{u} - \dot{D}\dot{u}\right) \qquad (35)\end{aligned}$$

From the differential equation $\ddot{u} + p\dot{u} + qu = 0$, we obtain:

$$\ddot{u} + p\dot{u} = -qu \tag{36}$$

From (31), we obtain:

$$\frac{\dot{D}}{D} = -p \tag{37}$$

Hence, we can simplify the expression for J_1 as follows:

$$\begin{aligned}
J_1 &= \frac{6u^3}{D^2}\left(\ddot{u} - \frac{\dot{D}}{D}\dot{u}\right) \\
&= \frac{6u^3}{D^2}\left(\ddot{u} + p\dot{u}\right) \qquad \text{using (37)} \\
&= \frac{6u^3}{D^2}(-qu) \qquad \text{using (36)} \\
&= \frac{-6qu^4}{D^2}
\end{aligned} \tag{38}$$

d)

We find an expression for J_2:

$$\begin{aligned}
\dot{J}_1 &= -6\frac{D^2\big(q(4u^3\dot{u}) + u^4\dot{q}\big) + qu^4(2D)\dot{D}}{D^4} \\
&= -\frac{6u^3}{D^2}\left(4q\dot{u} + \dot{q}u + 2pqu\right) \qquad \text{using (37)} \\
J_1' &= \frac{\dot{J}_1}{\dot{z}} \\
&= -\frac{6u^5}{D^3}\left(4q\dot{u} + \dot{q}u + 2pqu\right) \\
J_2 &= J_1' - \frac{2}{3}wJ_1 \\
&= -\frac{6u^5}{D^3}\left(4q\dot{u} + \dot{q}u + 2pqu\right) - \frac{2}{3}\cdot\frac{6u\dot{u}}{D}\cdot\frac{-6qu^4}{D^2} \\
&= -\frac{6u^5}{D^3}\left(4q\dot{u} + \dot{q}u + 2pqu\right) + \frac{24qu^5\dot{u}}{D^3} \\
&= \frac{6u^5}{D^3}\left(-4q\dot{u} - \dot{q}u - 2pqu + 4\dot{u}q\right) \\
&= \frac{6u^6}{D^3}\left(-2pq - \dot{q}\right)
\end{aligned} \tag{39}$$

Firstly, we use (25) to find J_3:

$$\begin{aligned}
J_3 &= -4(J_1)^2 \\
&= -4\left(\frac{36q^2u^8}{D^4}\right) \\
&= \frac{-144q^2u^8}{D^4}
\end{aligned} \tag{40}$$

Secondly, we evaluate J_3 using its definition (22). Firstly, we note that:

$$\frac{d}{dt}(-2pq - \dot{q}) = -2(p\dot{q} + \dot{p}q) - \ddot{q} \tag{41}$$

$$\begin{aligned} \frac{d}{dt}(6u^6 D^{-3}) &= 6\left\{\frac{-3u^6\dot{D}}{D^4} + \frac{6u^5\dot{u}}{D^3}\right\} \\ &= \frac{6u^5}{D^3}\left(-3up + 6\dot{u}\right) \end{aligned}$$

Hence:

$$\begin{aligned} \dot{J}_2 &= \frac{6u^6}{D^3}(-2p\dot{q} - 2\dot{p}q - \ddot{q}) + (-2pq - \dot{q})\frac{6u^5}{D^3}\left(-3up + 6\dot{u}\right) \\ &= \frac{6u^5}{D^3}(-2up\dot{q} - 2u\dot{p}q - u\ddot{q} + 6up^2q - 12\dot{u}pq + 3up\dot{q} - 6\dot{u}q) \\ J_3 &= \frac{\dot{J}_2}{\dot{z}} - wJ_2 \\ &= \frac{6u^7}{D^4}(up\dot{q} - 2u\dot{p}q - u\ddot{q} + 6up^2q - 12\dot{u}pq - 6\dot{u}\dot{q}) - \frac{6u\dot{u}}{D}\frac{6u^6}{D^3}(-2pq - \dot{q}) \\ &= \frac{6u^7}{D^4}(up\dot{q} - 2u\dot{p}q - u\ddot{q} + 6up^2q - 12\dot{u}pq - 6\dot{u}\dot{q} + 12\dot{u}pq + 6\dot{u}\dot{q}) \\ &= \frac{6u^8}{D^4}(p\dot{q} - 2\dot{p}q - \ddot{q} + 6p^2q) \end{aligned} \tag{42}$$

e)

Combining (40) and (42), we obtain a single differential constraint on p and q:

$$p\dot{q} - 2p\dot{q} - \ddot{q} + 6p^2q = -24q^2 \tag{43}$$

f)

By setting $p(t) = 0$ in (43), we obtain:

$$\ddot{q} = 24q^2 \tag{44}$$

$$2\ddot{q}\dot{q} = 48q^2\dot{q}$$

$$(\dot{q})^2 = 16q^3 + \frac{c_1}{16} \tag{45}$$

Let $q = \frac{r}{4}$. Then $(\dot{r})^2 = 4r^3 + c_1$, which is solved by:

$$r = \wp(z - c_2; 0, c_1) \tag{46}$$

Hence, the solution of the differential constraint is:

$$q = \frac{1}{4}\wp(z - c_2; 0, c_1) \tag{47}$$

where c_1, c_2 are constants of integration.

Question 3

a)

The complete elliptic integrals:

$$w_1(x) = K(\sqrt{x}) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - x\sin^2 t}} \tag{48}$$

and

$$w_2(x) = E(\sqrt{x}) = \int_0^{\pi/2} \sqrt{1 - x\sin^2 t}\, dt \tag{49}$$

when differentiated under the integral sign with respect to x become, respectively:

$$\frac{dw_1}{dx} = \int_0^{\pi/2} \frac{\sin^2 t}{2(1 - x\sin^2 t)^{3/2}}\, dt \tag{50}$$

$$\frac{dw_2}{dx} = \int_0^{\pi/2} \frac{-\sin^2 t}{2\sqrt{1 - x\sin^2 t}} \tag{51}$$

The first identity can be shown using (51):

$$\begin{aligned}
\frac{w_2 - w_1}{2x} &= \frac{1}{2x}\int_0^{\pi/2} \left(\sqrt{1 - x\sin^2 t} - \frac{1}{\sqrt{1 - x\sin^2 t}}\right) dt \\
&= \int_0^{\pi/2} \frac{(1 - x\sin^2 t) - 1}{2x\sqrt{1 - x\sin^2 t}}\, dt \\
&= \int_0^{\pi/2} \frac{-\sin^2 t}{2\sqrt{1 - x\sin^2 t}} \\
&= \frac{dw_2}{dx}
\end{aligned} \tag{52}$$

To prove the second identity, we will first find $\frac{dK}{dk}$:

$$K(k) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2\sin^2 t}}\, dt \tag{53}$$

$$\frac{dK}{dk} = \int_0^{\pi/2} \frac{k\sin^2 t}{(1 - k^2\sin^2 t)^{3/2}}\, dt \tag{54}$$

Let $u = \displaystyle\int_0^t \frac{dv}{\sqrt{1 - k^2\sin^2 v}}$. Then $\operatorname{sn} u = \sin t$. Noting that $\dfrac{du}{dt} = \dfrac{1}{\sqrt{1 - k^2\sin^2 t}}$ by the Fundamental Theorem of Calculus, and that $\operatorname{sn} K = \sin\frac{\pi}{2}$:

$$\begin{aligned}
\frac{dK}{dk} &= \int_0^{\pi/2} \frac{k\sin^2 t}{1 - k^2\sin^2 t} \cdot \frac{dt}{\sqrt{1 - k^2\sin^2 t}} \\
&= \int_0^K \frac{k\operatorname{sn}^2 u}{1 - k^2\operatorname{sn}^2 u}\, du \\
&= k\int_0^K \frac{\operatorname{sn}^2 u}{\operatorname{dn}^2 u}\, du
\end{aligned} \tag{55}$$

Following the suggestion in Whittaker and Watson (1963) §22.736, we prove the following identity:

$$\int \operatorname{dn}^2 u\, du = k'^2 u + k^2 \operatorname{sn} u \operatorname{cd} u + k^2 k'^2 \int \operatorname{sd}^2 u\, du \tag{56}$$

by differentiating both sides. First, we note the following:

$$\begin{aligned}
\frac{d}{du}\operatorname{cd} u &= \frac{\operatorname{dn} u(-\operatorname{sn} u \operatorname{dn} u) - \operatorname{cn} u(-k^2 \operatorname{sn} u \operatorname{cn} u)}{\operatorname{dn}^2 u} \\
&= \frac{\operatorname{sn} u(k^2 \operatorname{cn}^2 u - \operatorname{dn}^2 u)}{\operatorname{dn}^2 u} \\
&= \frac{-k'^2 \operatorname{sn} u}{\operatorname{dn}^2 u} \qquad (57) \\
\frac{d}{du}\operatorname{sn} u \operatorname{cd} u &= \operatorname{sn} u \left(\frac{-k'^2 \operatorname{sn} u}{\operatorname{dn}^2 u}\right) + \frac{\operatorname{cn} u}{\operatorname{dn} u}(\operatorname{cn} u \operatorname{dn} u) \\
&= \frac{-k'^2 \operatorname{sn}^2 u}{\operatorname{dn}^2 u} + \operatorname{cn}^2 u \qquad (58)
\end{aligned}$$

Therefore:

$$\begin{aligned}
&\frac{d}{du}(k'^2 u + k^2 \operatorname{sn} u \operatorname{cd} u + k^2 k'^2 \int \operatorname{sd}^2 u\, du) \\
&= k'^2 - \frac{k^2 k'^2 \operatorname{sn}^2 u}{\operatorname{dn}^2 u} + k^2 \operatorname{cn}^2 u + k^2 k'^2 \frac{\operatorname{sn}^2 u}{\operatorname{dn}^2 u} \\
&= 1 - k^2 + k^2 \operatorname{cn}^2 u \\
&= 1 - k^2 \operatorname{sn}^2 u \\
&= \operatorname{dn}^2 u \\
&= \frac{d}{du} \int \operatorname{dn}^2 u\, du \qquad (59)
\end{aligned}$$

Using (56), we obtain:

$$\begin{aligned}
\int \operatorname{dn}^2 u\, du - k'^2 u - k^2 \operatorname{sn} u \operatorname{cd} u &= k^2 k'^2 \int \operatorname{sd}^2 u\, du \\
\frac{1}{kk'^2} \int \operatorname{dn}^2 u\, du - \frac{u}{k} - \frac{k \operatorname{sn} u \operatorname{cd} u}{k'^2} &= k \int \operatorname{sd}^2 u\, du \\
\frac{1}{kk'^2} \int_0^K \operatorname{dn}^2 u\, du - \left[\frac{u}{k}\right]_0^K - \left[\frac{k \operatorname{sn} u \operatorname{cd} u}{k'^2}\right]_0^K &= k \int_0^K \operatorname{sd}^2 u\, du \\
\frac{1}{kk'^2} \int_0^K \operatorname{dn}^2 u\, du - \frac{K}{k} &= k \int_0^K \operatorname{sd}^2 u\, du \qquad (60)
\end{aligned}$$

since $\operatorname{sn} 0 = 0$ and $\operatorname{cd} K = 0$.

Continuing from (55), and using (60), we obtain:

$$\frac{dK}{dk} = \frac{E}{kk'^2} - \frac{K}{k} \tag{61}$$

since $E = \int_0^K \operatorname{dn}^2 u\, du = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 t}\, dt$ by the same substitution as before.

We can now prove the second identity, using (61):

$$\begin{aligned}\frac{dw_1}{dx} &= \frac{dK(\sqrt{x})}{d\sqrt{x}}\cdot\frac{d\sqrt{x}}{dx}\\ &= \left(\frac{E(\sqrt{x})}{\sqrt{x}(1-x)} - \frac{K(\sqrt{x}}{\sqrt{x}}\right)\cdot\frac{1}{2\sqrt{x}}\\ &= \frac{w_2}{2x(1-x)} - \frac{w_1}{2x} \qquad (62)\end{aligned}$$

We shall now show that w_1 and w_2 satisfy hypergeometric differential equations.

Firstly, we note that:

$$\begin{aligned}\frac{d}{dx}(w_2-w_1) &= \frac{w_2-w_1}{2x} - \frac{w_2-(1-x)w_1}{2x(1-x)}\\ &= \frac{-w_2}{2(1-x)}\\ \frac{d^2w_2}{dx^2} &= \frac{2x\left(\frac{-w}{2(1-x)}\right)-(w_2-w_1)(2)}{4x^2}\\ &= \frac{-xw_2-2(1-x)(w_2-w_1)}{4x^2(1-x)}\\ &= \frac{(x-2)w_2+2(1-k)w_1}{4x^2(1-x)} \qquad (63)\\ \frac{d}{dx}(w_2-(1-x)w_1) &= \frac{w_2-w_1}{2x} - (1-x)\left(\frac{w_2}{2x(1-x)}-\frac{w_1}{2x}\right) - w_1(-1)\\ &= \frac{w_1}{2}\\ \frac{d^2w_1}{dx^2} &= \frac{2x(1-x)\frac{w_1}{2} - \big(w_2-(1-x)w_1\big)(2-4x)}{4x^2(1-x^2}\\ &= \frac{w_1(x-1)(3x-2)+w_2(4x-2)}{4x^2(1-x)^2} \qquad (64)\end{aligned}$$

Substitute w_2 into the hypergeometric differential equation to obtain:

$$\begin{aligned}x(1-x)w_2'' + \big(\gamma-(1+\alpha+\beta)x\big)w_2' - \alpha\beta w_2 &= 0\\ \frac{-(2+x(1+2\alpha+4\alpha\beta+2\beta)-2\gamma)w_2 + 2(1+x(\alpha+\beta)-\gamma)w_1}{4x} &= 0 \qquad (65)\end{aligned}$$

For this equation to hold, the coefficients of w_1 and w_2 must be equal to zero:

$$\begin{aligned}-(2+x(1+2\alpha+4\alpha\beta+2\beta)-2\gamma) &= 0\\ x &= \frac{2\gamma-2}{1+2\alpha+4\alpha\beta+2\beta} \qquad (66)\\ 2(1+x(\alpha+\beta)-\gamma) &= 0\\ x &= \frac{\gamma-1}{\alpha+\beta} \qquad (67)\end{aligned}$$

Solving (66) and (67) simultaneously yields:

$$(1+4\alpha\beta)(\gamma-1) = 0 \qquad (68)$$

Hence, w_2 satisfies the hypergeometric differential equation for $\gamma = 1$ or $\alpha\beta = -\frac{1}{4}$.
Substitute w_1 into the hypergeometric differential equation to obtain:

$$\begin{aligned} x(1-x)w_1'' + \big(\gamma - (1+\alpha+\beta)x\big)w_1' - \alpha\beta w_1 &= 0 \\ \frac{2(1+x(-1+\alpha+\beta)-\gamma)w_2 + (x-1)(2+x(-1+2\alpha-4\alpha\beta+2\beta)-2\gamma)w_1}{4x(x-1)} &= 0 \end{aligned} \tag{69}$$

For this equation to hold, the coefficients of w_1 and w_2 must be equal to zero:

$$\begin{aligned} 2(1+x(-1+\alpha+\beta)-\gamma) &= 0 \\ x &= \frac{\gamma-1}{\alpha+\beta-1} \end{aligned} \tag{70}$$

$$\begin{aligned} (x-1)(2+x(-1+2\alpha-4\alpha\beta+2\beta)-2\gamma) &= 0 \\ x &= \frac{2\gamma-2}{-1+2\alpha-4\alpha\beta+2\beta} \qquad \text{or } x = 1 \end{aligned} \tag{71}$$

Solving (70) and (71) simultaneously yields:

$$(1-4\alpha\beta)(1-\gamma) = 0 \tag{72}$$

Hence, w_1 satisfies the hypergeometric differential equation for $x = 1$ or $\gamma = 1$ or $\alpha\beta = \frac{1}{4}$.

b)

We use the integral formula given in the lecture notes:

$$F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\beta)\Gamma(\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}\,dt \tag{73}$$

Firstly:

$$\begin{aligned} F\left(\frac{1}{2},\frac{1}{2};1;k^2\right) &= \frac{\Gamma(1)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_0^1 t^{-1/2}(1-t)^{-1/2}(1-k^2t)^{-1/2}\,dt \\ &= \frac{1}{\pi}\int_0^1 \frac{dt}{\sqrt{t}\sqrt{1-t}\sqrt{1-k^2t}}, \qquad t = u^2 \\ &= \frac{1}{\pi}\int_0^1 \frac{2u}{\sqrt{u^2}\sqrt{1-u^2}\sqrt{1-k^2u^2}}\,du \\ &= \frac{2}{\pi}\int_0^1 \frac{du}{\sqrt{1-u^2}\sqrt{1-k^2u^2}} \\ &= \frac{2}{\pi}K(k) \\ K(k) &= \frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;k^2\right) \end{aligned} \tag{74}$$

Furthermore:

$$\begin{aligned}
F\left(-\frac{1}{2},\frac{1}{2};1;k^2\right) &= \frac{\Gamma(1)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_0^1 t^{-1/2}(1-t)^{-1/2}(1-k^2t)^{1/2}\,dt \\
&= \frac{1}{\pi}\int_0^1 \frac{\sqrt{1-k^2t}}{\sqrt{t}\sqrt{1-t}}\,dt, \qquad t=u^2 \\
&= \frac{1}{\pi}\int_0^1 \frac{\sqrt{1-k^2u^2}\,2u}{\sqrt{u^2}\sqrt{1-u^2}}\,du \\
&= \frac{2}{\pi}\int_0^1 \frac{\sqrt{1-k^2u^2}}{\sqrt{1-u^2}}\,du \\
&= \frac{2}{\pi}E(k) \\
E(k) &= \frac{\pi}{2}F\left(-\frac{1}{2},\frac{1}{2};1;k^2\right) \qquad (75)
\end{aligned}$$