THE UNIVERSITY OF SYDNEY

COMPLEX ANALYSIS (ADV.)

MATH3964	Complex Analysis	2006

Assignment 2 *Due Monday, 23 October, 2006*

This assignment is worth 15% of the course assessment.

1. (a) Given that $w(z)$ satisfies the Riccati equation,

$$w' = P(z)w^2 + Q(z)w + R(z), \qquad ' = d/dz,$$

find an expression for w'' as a cubic polynomial in w.

(b) Find all Riccati equations,

$$w' = P(z)w^2 + Q(z)w + R(z), \qquad ' = d/dz,$$

that are compatible with the following differential equation:

$$w'' = \frac{1}{2w}(w')^2 + \frac{3w^3}{2} + 4zw^2 + 2(z^2 - \alpha)w - \frac{\beta^2}{2w}.$$

(This, by the way, is the Painlevé-IV equation.) It is recommended that you first show that

$$P(z) = \epsilon_1, \qquad R(z) = \epsilon_2\beta,$$

where ϵ_1 and ϵ_2 denote ± 1 independently. Thereafter, you only need to find $Q(z)$. There is a constraint on the parameters α and β, which you must find, for this problem to have a solution (express β in terms of α, ϵ_1 and ϵ_2).

(c) Express the solution of the Riccati equation in terms of the solutions of a linear differential equation of the form $u'' + f(z)u = 0$.

2. Painlevé analysis is able to discover interesting integrable equations without showing how to actually integrate them. One such equation is

$$w''' = 2ww'' - 3(w')^2, \qquad ' = d/dz,$$

which was discovered by Chazy (1909), and is now called the Chazy-III equation because it is the third equation on a list of 13 Painlevé-type equations derived by Chazy two years later.

(a) Show that the Chazy-III equation can be written as $J_3 + 4(J_1)^2 = 0$, where

$$J_1 = \frac{dw}{dz} - \frac{w^2}{6}, \qquad J_2 = \frac{dJ_1}{dz} - \frac{2}{3}wJ_1, \qquad J_3 = \frac{dJ_2}{dz} - wJ_2.$$

(b) Suppose that $u(t)$ satisfies a linear differential equation of the form,

$$\ddot{u} + p(t)\dot{u} + q(t)u = 0,$$

where the dot denotes d/dt. Let $u(t)$ and $u_1(t)$ be any two linearly independent solutions. Define the Wronskian, $D = u\dot{u}_1 - u_1\dot{u}$. Show that

$$D = k \exp\Big\{ - \int p(t)\, dt \Big\},$$

where k is some nonzero constant.

(c) Attempt to solve the Chazy-III equation by parametric equations of the form,

$$z = \frac{u_1}{u}, \qquad w(z) = \frac{6u\dot{u}}{D}.$$

Derivatives of w are calculated by the chain rule, for example, $w'(z) = \dot{w}/\dot{z}$. Calculate $w'(z)$ and then J_1 in terms of t.

(d) Calculate J_2 and J_3 in terms of t. (Note that these are easier to calculate than the derivatives of w.)

(e) Obtain a single differential constraint on $p(t)$ and $q(t)$ such that the postulated solution of Chazy-III is valid.

(f) Without loss of generality, set $p(t) = 0$. Solve the differential constraint for $q(t)$ in terms of Weierstrass elliptic functions.

3. (a) Differentiate the complete elliptic integrals,

$$w_1(x) := K(\sqrt{x}) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - x\sin^2 t}},$$

$$w_2(x) := E(\sqrt{x}) = \int_0^{\pi/2} \sqrt{1 - x\sin^2 t}\, dt,$$

having modulus $k = \sqrt{x}$, under the integral sign with respect to x and deduce the identities,

$$\frac{dw_2}{dx} = \frac{w_2 - w_1}{2x}.$$

$$\frac{dw_1}{dx} = \frac{w_2}{2x(1-x)} - \frac{w_1}{2x}.$$

Deduce that $w_1(x)$ and $w_2(x)$ each satisfy particular hypergeometric differential equations.

(b) Derive the exact formula,

$$K(k) = \tfrac{1}{2}\pi\, F\big(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2\big),$$

and a similar formula for $E(k)$.