MATH3964 Lecture Week 11-2

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1 Transformations of Fuchsian DEs

- $\bar{w} = (z - z_1)^\lambda w$
- Raises all elements at z_1 by λ, lowers all exponents at ∞ by λ
- Exponents at other regular singular points are not affected
- The DE simplifies if all finite regular singular points have a zero exponent
- Mobius transformation in z: $\bar{z} = \dfrac{az + b}{cz + d}$
- This moves the singular points (including ∞); exponents are not affected
- The DE simplifies if any 3 regular singlar points are moved to $\bar{z} = 0, 1, \infty$
- There are 6 permutations: $z \to z, 1 - z, \frac{1}{z}, \frac{z}{z-1}, \frac{1}{1-z}, 1 - \frac{1}{z}$
- If z_0 is an ordinary nonsingular point, then its indicial equation can still be calculated and its exponents are $\rho = 0, 1, 2, \ldots, n - 1$ (DE order n) and there are no logs. The converse is also true.
- If a regular singular points z_0 has exponents $\alpha, \alpha + 1, \alpha + 2, \ldots, \alpha + n - 1$ and is free of logs, then it is removable by the change of variable $\bar{w} = (z - z_0)^{-\alpha} w$
- If the exponents at z_0 are distinct non-negative integers, and there are no logs, then $w(z)$ is analytic at z_0. Such a regular singlar point is called an apaprent singularity. In special circumstances, they can be removed from the DE by a change of variables.
- The quadratic transformation $z = \bar{z}^2$ doubles the exponents at 0 and ∞. Other singular points z_0 become two singular points $\pm\sqrt{z_0}$ with the same exponents.

2 Case of 3 regular singular points

- Put them at $z = 0, 1, \infty$
- At $z = 0, 1$, make one of the exponents zero
- Standard notation:

- $z = 0$: $\rho = 0, 1-\gamma$
- $z = 1$: $\rho = 0, \gamma - \alpha - \beta$
- $z = \infty$: $\rho = \alpha, \beta$

- These exponents uniquely determine the DE. We get the hypergeometric DE: $z(1-z)w'' + \big(\gamma - (1+\alpha+\beta)z\big)w' - \alpha\beta w = 0$
- The fundamental solution is the **hypergeometric function** associated with $z = 0$ and $\rho = 0$. We must assume that γ is not zero or a negative integer.
- Substitute $w = \sum_{n=0}^{\infty} a_n z^n$ with $a_0 = 1$ into the DE. The recurrence relation for the a_n is $a_{n+1} = \dfrac{(\alpha+n)(\beta+n)}{(\gamma+n)(n+1)} a_n$.
- Hence, $a_n = \dfrac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}$ where $(\alpha)_n := \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$ (Pochhammer symbol), $n = 0, 1, 2, \ldots$, $\alpha \in \mathbb{C}$, $\alpha \neq 0, -1, -2, \ldots$
- Once solution is $w = F(\alpha, \beta; \gamma; z) := \sum_{n=0}^{\infty} \dfrac{(\alpha)_n(\beta)_n}{(\gamma)_n n!} z^n$
- To get a linearly independent solution, use $\rho = 1 - \gamma$ at $z = 0$. Now γ cannot be any integer.
- Substitute $w = z^{1-\gamma} \sum_{z=0}^{\infty} a_n z^n$, $b_0 = 1$. Get $b_{n+1} = \dfrac{(1+\alpha-\gamma+n)(1+\beta-\gamma+n)}{(2-\gamma+n)(n+1)} b_n$. The result is $w = z^{1-\gamma} F(1+\alpha-\gamma, 1+\beta-\gamma; z-\gamma; z)$
- The general solution of the hypergeometric DE where $\gamma \notin \mathbb{Z}$ is $w = C_1 F(\alpha, \beta; \gamma; z) + C_2 z^{1-\gamma} F(1+\alpha-\gamma, 1+\beta-\gamma; z-\gamma; z)$
- Another name for F is ${}_2F_1$.
- The generalised hypergeometric function is $F(\alpha_1, \alpha_2, \ldots, \alpha_p; \gamma_1, \gamma_2, \ldots, \gamma_q; z)$; also written as $F\left(\begin{matrix} \alpha_1, \alpha_2, \ldots, \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q; \end{matrix}\; z\right)$
- This is defined by $\sum_{n=0}^{\infty} \dfrac{(\alpha_1)_n(\alpha_2)_n \cdots (\alpha_p)_n}{(\gamma_1)_n(\gamma_2)_n \cdots (\gamma_q)_n n!} z^n$
- This converges to an entire function of z when $p \leq q$. It converges absolutely to an analytic function in the open unit disc $|z| < 1$ is $p = q + 1$. This includes ${}_2F_1$.