MATH3964 Lecture Week 11-3

Enoch Lau

1 Identities for the hypergeometric function

- Collect the information about regular singular points and their exponents in the following symbol (which can be defined for any Fuchsian-type DE):

$$w = P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha & z \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{matrix}\right\} \qquad \text{(Riemann } P\text{-function)}$$

- This w can be regarded as a solution of the hypergeometric DE (a function of z with 9 parameters). The two constants of integration are not specified. We have the choice of using this symbol for the general solution or a particular solution.
- Some hypergeometric identities can be read straight off the P-function.
- Let $P(\ldots)$ denote $F(\alpha, \beta; \gamma; z)$:

$$\begin{aligned} w &= P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha & z \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{matrix}\right\} \\ &= (1-z)^{\gamma-\alpha-\beta} P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & \alpha+\beta-\gamma & \gamma-\beta & z \\ 1-\gamma & 0 & \gamma-\alpha & \end{matrix}\right\} \\ F(\alpha,\beta;\gamma;z) &= (1-z)^{\gamma-\alpha-\beta}\{C_1 F(\gamma-\alpha,\gamma-\beta;\gamma;z) + C_2 z^{1-\gamma} F(\cdots)\} \end{aligned}$$

- At $z = 0$, LHS $= 1 + \dfrac{\alpha\beta}{\gamma} z + \ldots$. On RHS, $C_1 = 1$ and $C_2 = 0$.
- Hence, we get the identity:

$$F(\alpha,\beta;\gamma;z) = (1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha,\gamma-\beta;\gamma;z)$$

- Linearly independent solutions:

$$\begin{aligned} w &= P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha & z \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{matrix}\right\} \\ &= z^{1-\gamma} P\left\{\begin{matrix} 0 & 1 & \infty & \\ \gamma-1 & 0 & 1+\alpha-\gamma & z \\ 0 & \gamma-\alpha-\beta & 1+\beta-\gamma & \end{matrix}\right\} \\ &= z^{1-\gamma} F(1+\alpha-\gamma, 1+\beta-\gamma; 2-\gamma; z) \end{aligned}$$

- $z \mapsto z, 1-z, \dfrac{1}{z}, \dfrac{z}{z-1}, \dfrac{1}{1-z}, 1-\dfrac{1}{z}$. Of these, $\dfrac{z}{z-1}$ is 0 when $z=0$. Hence, we get an identity involving argument $\dfrac{z}{z-1}$. $\dfrac{z}{z-1}$ preserves $z=0$ and swaps $z=1$ and $z=\infty$.
- By replacing $z \mapsto \dfrac{z}{z-1}$, we can swap the second and third columns:

$$\begin{aligned}
w &= P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha & z \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{matrix}\right\} \\
&= P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & \alpha & 0 & \dfrac{z}{z-1} \\ 1-\gamma & \beta & \gamma-\alpha-\beta & \end{matrix}\right\} \\
&= \left(1-\frac{z}{z-1}\right)^{\alpha} P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha & \dfrac{z}{z-1} \\ 1-\gamma & \beta-\alpha & \gamma-\beta & \end{matrix}\right\} \\
&= (1-z)^{-\alpha}\left\{C_1 F(\alpha, \gamma-\beta; \gamma; \frac{z}{z-1}) + C_2\left(\frac{z}{z-1}\right)^{1-\gamma} F(\ldots)\right\}
\end{aligned}$$

- LHS $= F(\alpha, \beta; \gamma; z) \Rightarrow C_1 = 1$ and $C_2 = 0$.
- $F(\alpha, \beta; \gamma; z) = (1-z)^{-\alpha} F\left(\alpha, \gamma-\beta; \gamma; \dfrac{z}{z-1}\right)$
- Swapping α and β gives $F(\alpha, \beta; \gamma; z) = (1-z)^{-\beta} F\left(\gamma-\alpha, \beta; \gamma; \frac{z}{z-1}\right)$
- Additional identities can be found after we calculate Gauss' formula for $F(\alpha, \beta; \gamma; 1)$.

2 Convergence of the hypergeometric series

- Ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{(\alpha+n)(\beta+n)}{(\gamma+n)(n+1)} z$$

  As $n \to \infty$, $\left|\frac{a_{n+1}}{a_n}\right| \to |z|$. Series converges absolutely for $|z| < 1$. It diverges for $|z| > 1$ unless it terminates (α or β is $0, -1, -2, -3, \ldots$).
- To handle $|z| = 1$, we need large-n behaviour of $(\alpha)_n$. This in turn depends on Stirling's asymptotic formula: $n! \sim n^n e^{-n}\sqrt{2\pi n}$ as $n \to \infty$, or more precisely, $\Gamma(z) \sim \sqrt{2\pi} z^{z-1/2} e^{-z}$ as $|z| \to \infty$ for $-\pi + \delta \leq \arg z \leq \pi - \delta$.

- Proving convergence:

$$
\begin{aligned}
\frac{(\alpha)_n}{(\gamma)_n} &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)}\frac{\Gamma(\alpha+n)}{\Gamma(\gamma+n)} \\
\frac{\Gamma(\alpha+n)}{\Gamma(\gamma+n)} &\sim \frac{\sqrt{2\pi}(n+\alpha)^{n+\alpha-1/2}e^{-n-\alpha}}{\sqrt{2\pi}(n+\gamma)^{n+\gamma-1/2}e^{-n-\gamma}} \\
&\sim e^{\gamma-\alpha}\frac{n^{n+\alpha-1/2}\left(1+\frac{\alpha}{n}\right)^n\left(1+\frac{\alpha}{n}\right)^{\alpha-1/2}}{n^{n+\gamma-1/2}\left(1+\frac{\gamma}{n}\right)^n\left(1+\frac{\gamma}{n}\right)^{\gamma-1/2}} \\
&\sim n^{\alpha-\gamma} \\
\frac{(\alpha)_n}{(\gamma)_n} &\sim \frac{\Gamma(\gamma)}{\Gamma(\alpha)}n^{\alpha-\gamma} \qquad \text{as } n\to\infty \\
\frac{(\beta)_n}{n!} &= \frac{(\beta)_n}{(1)_n} \\
&\sim \frac{1}{\Gamma(\beta)}n^{\beta-1}
\end{aligned}
$$

- Hence, $\sum \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}z^n$ is comparable to $\sum \frac{z^n}{n^{\gamma-\alpha-\beta+1}}$:
  - $\mathrm{Re}\,(\gamma-\alpha-\beta) > 0 \Rightarrow$ series converges absolutely and uniformly on the closed disc $|z| \leq 1$.
  - $-1 < \mathrm{Re}\,(\gamma-\alpha-\beta) \leq 0 \Rightarrow$ series converges conditionally on circle $|z| = 1$ except at $z = 1$, where it diverges.
  - $\mathrm{Re}\,(\gamma-\alpha-\beta) \leq -1 \Rightarrow$ series diverges on the circle $|z| = 1$, terminating cases excepted.
- In particular the series for $F(\alpha,\beta;\gamma;1)$ converges (absolutely) iff $\mathrm{Re}\,(\gamma-\alpha-\beta) > 0$ unless it terminates (that is, it is a polynomial).