MATH3964 Summaries (Part 1)

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1 Definitions

- A subset $U \subseteq \mathbb{C}$ is **open** if $\forall\, a \in U$, $\exists\, r > 0$ such that $U_{a,r} \subseteq U$.
- A subset $D \subseteq \mathbb{C}$ is **closed** if its complement $\mathbb{C}\backslash D$ is open.
- A set is **compact** if it is closed and bounded.
- $\lim_{z\to a} f(z) = b$ if $\forall\, \epsilon > 0$, $\exists\, \delta$ such that $0 < |z-a| < \delta \Rightarrow |f(z) - b| < \epsilon$.
- f is **continuous** at a if $\lim_{z\to a} f(z) = f(a)$.
- If the limit $\dfrac{df(z)}{dz} = \lim\limits_{h\to 0} \dfrac{f(z+h) - f(z)}{h}$ exists, then f is **complex differentiable** at z.
- If f is complex differentiable at all points z in a non-empty connected open set U, f is **holomorphic** or **analytic** on U.
- An **entire function** is a function that is analytic throughout all of $\mathbb{C}$.
- A **singularity** is where a function fails to be analytic.

2 Power series

- Consider the power series $\sum\limits_{n=0}^{\infty} a_n (z-z_0)^n$, for $a_n \in \mathbb{C}$ and $z_0 \in \mathbb{C}$.
- The **radius of convergence** of the power series is:

$$R = \liminf_{n\to\infty} |a_n|^{-\frac{1}{n}}$$

- If $R > 0$ or $R = +\infty$, the series converges absolutely on the open disc $|z - z_0| < R$.
- If $0 \leq R < \infty$, the series diverges for $|z - z_0| > R$.
- The **circle of convergence** is $|z - z_0| = R$, and the series may converge absolutely, conditionally or diverge.

- If the series converges absolutely at one point z on $|z - z_0| = R$, it will converge absolutely and uniformly on the closed disc $|z - z_0| \leq R$.
- Every power series with $R > 0$ or $R = +\infty$ converges to an analytic function on the open disc of convergence:

$$\begin{aligned} f'(z) &= \sum_{n=1}^{\infty} n\, a_n (z - z_0)^{n-1} \\ f''(z) &= \sum_{n=2}^{\infty} n(n-1)\, a_n (z - z_0)^{n-2} \end{aligned}$$

- Some common Taylor series expansions:

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \\ \sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \\ \cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \\ e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \end{aligned}$$

3 Contours and integrals

- A **contour** is any curve in the complex plane with an arrow or orientation.
- A **closed curve** has $s(a) = s(b)$.
- A curve is a **simple arc** or a **Jordan arc** if it does not intersect with itself, that is, $s(t)$ is 1-1 on $[a, b]$.
- A curve is a **simple closed curve** or a **Jordan curve** if it is a simple arc and it is closed.
- A curve is **positively oriented** if the inside is on the left when C is followed in the direction of the arrow.
- A curve $s(t) = x(t) + iy(t)$ is **rectifiable**, or of finite arc length, if $x(t)$ and $y(t)$ are of bounded variation.
- A curve is **smooth** if $s'(t)$ is continous and non-zero on $[a, b]$. A curve is **piecewise smooth** if it has a finite number of smooth arcs joined end to end.

- Given a function $f : U \to \mathbb{C}$ and a curve $\mathscr{C}$ in U given by $s : [a, b] \to U$, if f is continuous, the **complex line integral** of f over $\mathscr{C}$ is given by:

$$\int_{\mathscr{C}} f(z)\, dz = \int_a^b f(s(t))s'(t)\, dt$$

- $s : [a, b] \to \mathbb{C}$ in the other direction is $r : [-b, -a] \to \mathbb{C}$:

$$\int_{-\mathscr{C}} f(z)\, dz = -\int_{\mathscr{C}} f(z)\, dz$$

- If $\mathscr{C} = \mathscr{C}_1 + \ldots + \mathscr{C}_n$:

$$\int_{\mathscr{C}} f(z)\, dz = \int_{\mathscr{C}_1} f(z)\, dz + \ldots + \int_{\mathscr{C}_n} f(z)\, dz$$

- The triangle inequality for integrals:

$$\left| \int_a^b f(t)\, dt \right| \leq \int_a^b |f(t)|\, dt$$

- For a curve $\mathscr{C}$ given by $s(t) = x(t) + iy(t)$, the **length of the curve** is:

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

- ***ML*-formula**:

$$\left| \int_{\mathscr{C}} f(z) dz \right| \leq ML$$

  where $|f(z)| \leq M \, \forall z \in \mathscr{C}$ and $L =$ length of $\mathscr{C}$.

4 Cauchy's theorem and Cauchy's formula

- Given:
  - An open set $U \subseteq \mathbb{C}$
  - $f : U \to \mathbb{C}$
  - $D \subseteq U$ with boundary $\mathscr{C}$
- **Cauchy's integral theorem**:

$$\oint_{\mathscr{C}} f(z)\, dz = 0$$

- **Cauchy's integral formula**:

$$\left(\left(\frac{d}{dz}\right)^n f\right)(a) = \frac{n!}{2\pi i} \oint_{\mathscr{C}} \frac{f(z)\, dz}{(z-a)^{(n+1)}}$$

  for some a in the interior of D

- Green's theorem:

$$\int_{\mathscr{C}} P\,dx + Q\,dx = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

- **Cauchy Riemann equations**:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}$$

  if $f(z) = u(z) + iv(z)$.

- **Maximum modulus principle**: If $M = \sup_{z \in \mathbb{C}} |f(z)|$, then $|f(z)| \leq M \, \forall z \in U$.

5 Cauchy's inequality

- Given:
  - f is analytic in an open set $G \subseteq \mathbb{C}$
  - $D_{a,R} = \{z : |z - a| \leq R\} \subseteq G$
  - $|f(z)| \leq M$ for $z \in \{z : |z - a| = R\}$
- **Cauchy's inequality**:

$$\left| f^{(k)}(w) \right| \leq \frac{k!MR}{(R - |w - a|)^{k+1}}$$

  for integers $k \geq 0$ and $w \in U_{a,R}$.

- In particular, for $w = a$:

$$\left| f^{(k)}(a) \right| \leq \frac{k!M}{R^k}$$

- **Liouville's theorem**: If $f(z)$ is analytic and bounded for all $z \in \mathbb{C}$, then f is constant.

6 Morera's theorem

- **Morera's theorem**: If $f : G \to \mathbb{C}$ is a continuous function such that $\int_T f(z)\,dz = 0$ for every triangle T in G, then f is analytic in G.

7 Weierstrass theorem

- A sequence $f_n(z)$ **converges uniformly** on a compact set K to $f(z)$ if $\forall \epsilon > 0$, $\exists N$ such that if $n > N$, then $|f_n(z) - f(z)| < \epsilon \, \forall z \in K$.
- A **region** is a non-empty open connected set in $\mathbb{C}$.
- **Weierstrass theorem**: If $f_n(z)$ is analytic in the region Ω and the sequence $\{f_n(z)\}$ converges to $f(z)$ in Ω uniformly on every compact subset of Ω, then $f(z)$ is analytic in Ω. Moreover, $f_n'(z)$ converges uniformly to $f'(z)$ on every compact subset of Ω.
- Corollary: If f_n is analytic on Ω, and $\sum_{n=1}^{\infty} f_n$ converges uniformly on compact subsets to f, then f is analytic on Ω and $f'(z) = \sum_{n=1}^{\infty} f_n'(z)$.
- If $F : G \times [a, \infty) \to \mathbb{C}$ is a continuous function such that $F(z,t)$ is analytic for every t. Then:
  1. For $b < \infty$, $f(z) = \int_a^b F(z,t)\, dt$ is analytic in G and $f'(z) = \int_a^b \frac{\partial F}{\partial z}(z,t)\, dt$.
  2. If $f(z) = \int_a^\infty F(z,t)\, dt$ is uniformly convergent on compact subsets of G then $f(z)$ is analytic in G and $f'(z) = \int_a^\infty \frac{\partial F}{\partial z}(z,t)\, dt$.

8 Eisenstein series

- The **upper half plane** is $\mathfrak{h} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.
- For a, b, c and d integers, $ad - bc = 1$, the multiplicative group of integral matrices of determinant 1 is:

$$\Gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

- A function $f : \mathfrak{h} \to \mathbb{C}$ is of **weight** $k \in \mathbb{Z}$ for Γ if $\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $z \in \mathfrak{h}$:

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = f(z)$$

- A function $f(z)$ is **periodic** if $\forall n \in \mathbb{N}$, $f(z+n) = f(z)$.
- The **Eisenstein series** of weight $k > 2$ is, for $\operatorname{Im} z > 0$:

$$G_k(z) = \sum_{(m,n) \neq (0,0)} \frac{1}{(mz+n)^k}$$

- It is analytic on $\mathfrak{h}$. It converges absolutely uniformly on compact subsets of $\mathfrak{h}$.