MATH3964 Summaries (Part 2)

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1 Laurent series

- Let f be analytic on $R < |z-a| < R'$. Then:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

  with the coefficients:

$$a_n = \frac{1}{2\pi i} \int_P \frac{f(z)\,dz}{(x-a)^{n+1}}$$

  where P denotes C (the inner contour) for $n < 0$ and C' (the outer contour) for $n \geq 0$.

- This is uniformly convergent in $R_1 \leq |z-a| \leq R_1'$, provided that $R < R_1 < R_1' < R'$.

2 Identity theorem

- Let G be a non-empty connected set in $\mathbb{C}$. Let $f : G \to \mathbb{C}$ be an analytic function and $\{z_k\}$ be a convergent sequence with limit $a \in G$. Then $f(z_k) = 0$ implies that $f(z) = 0 \, \forall z \in G$.
- **Identity theorem**: If f and g are analytic on G, $\{g_k\} \to a \in G$, and $f(z_k) = g(z_k) \, \forall k \in \mathbb{N}$, then $f(z) = g(z) \, \forall z \in G$.

3 Singularities and analytic functions

- An **entire function** is a function that is analytic, or can be analytically continued, to the whole complex plane. Its radius of convergence is ∞.
- A point $z_0 \in \mathbb{C}$ is a **singularity** of $f(z)$, if $f(z)$ cannot be continued to an open set containing z_0.
- A power series with a radius of convergence $0 < R < \infty$ converges to an analytic function of z inside the circle of convergence $|z - z_0| < R$ and has at least one singularity on the boundary.

- There is a **removable singularity** at z_0 if $f(z)$ is analytic and single-valued in a deleted neighbourhood of z_0 and $\lim_{z \to z_0} (z - z_0) f(z) = 0$, and there exists a unique complex number $f(z_0)$ that makes $f(z)$ analytic at z_0.

- Analyticity of derivative: If $f(z)$ is analytic in D and $a \in D$, then:

$$g(z) = \frac{f(z) - f(a)}{z - a}$$

  is analytic in D, and $g(a) = f'(a)$.

- $f(z)$ is analytic at $z = \infty$ if $g(z) = f\left(\frac{1}{z}\right)$ is analytic (has a removable singularity) at $z = 0$. Similarly, $f(z)$ has a pole, essential singularity, branch point, etc, at $z = \infty$ if $g(z)$ has such a singularity at $z = 0$.

4 Poles

- A point a is a **pole of order *n*** of $f(z)$, if $f(z) = \dfrac{k(z)}{(z-a)^n}$, for $z \in D \backslash \{a\}$, $D = \{z : |z - a| < R\}$, if $n > 0$, $k(a) \neq 0$ and k is analytic in D.
  - A pole is **simple** when $n = 1$.
  - A pole is **double** when $n = 2$.

- A function $f(z)$ is **meromorphic** in D, if $f(z)$ is analytic in D with the exception of a countable number of poles. The limit points of poles must be on the boundary of D or at ∞.

- If $f(z)$ is single-valued and analytic in a deleted neighbourhood of z_0 and if $|f(z)| \to \infty$ on all paths as $z \to z_0$, then the singularity of $f(z)$ at $z = z_0$ must be a pole. Conversely, if $f(z)$ has a pole at $z = z_0$, then $|f(z)| \to \infty$ as $z \to z_0$ on all paths.

5 Essential singularities

- A point $z_0 \in \mathbb{C}$ is an **essential singularity** of $f(z)$, if $f(z)$ is single-valued near $z = z_0$, not analytic at $z = z_0$, and $z = z_0$ is not a pole. If $f(z)$ is analytic in a deleted neighbourhood of z_0, then z_0 is an **isolated essential singularity**.

- A **natural barrier** is a continuum of essential singularities.

- If z_0 is a limit point of a-points of $f(z)$ (points z_n where $f(z_n) = a$), then z_0 is an essential singularity of $f(z)$.

- **Grand Ricard Theorem**: In any deleted neighbourhood of an isolated essential singularity of $f(z)$, $f(z)$ takes on every complex value a, with at most one exception, infinitely many times.

6 Branch points

- A point $z_0 \in \mathbb{C}$ is a **branch point** of $f(z)$, if $f(z)$ can be analytically continued to a multi-valued function in every deleted neighbourhood of z_0. Alternatively, $f(z)$ remains single-valued on a multi-sheeted covering of the complex plane called a **Riemann surface**.
- When you augment the finite complex plane by a single point at infinity, the **extended complex plane** has the topology of a **Riemann sphere**.

7 Zeros

- A point a is a **zero of order *m*** of $g(z)$, if $g(z) = (z-a)^m h(z)$, $z \in D = \{z : 0 \leq |z-a| < R\}$, $m > 0$, $h(a) \neq 0$ and h is analytic in D.
- Suppose a is a zero of order n of g, and p is analytic at a, $p(a) \neq 0$. Then, a is a pole of order n of $\frac{p}{g}$ in some disc D.

8 Residues

- The **residue** of f at a, which is a pole of order n, is:

$$\text{Res}(f, a) = \lim_{z \to a} \frac{1}{(n-1)!} \left(\frac{d}{dz} \right)^{n-1} \big((z-a)^n f(z) \big)$$

- If a is a pole of order $n \geq 1$ of f, then for a positively-oriented circle C of sufficiently small radius:

$$\int_C f(z)\, dz = 2\pi i \,\text{Res}(f, a)$$

- **Residue theorem**: Suppose f is analytic on a region D except for a finite number of poles at $a_1, \ldots, a_l$ in the interior of D. For the boundary of D:

$$\int_C f(z)\, dz = 2\pi i \sum_{j=1}^{l} \text{Res}(f, a_j)$$

9 Argument principle

- The **order of *f* at *a*** is:

$$O(f, a) = \begin{cases} k & \text{if } f \text{ has a zero of order } k \geq 1 \\ 0 & \text{if } f \text{ is analytic at } a,\ f(a) \neq 0 \\ -k & \text{if } f \text{ has a pole of order } k \geq 1 \end{cases}$$

- **Argument principle**: If f is analytic and non-zero on $G\backslash A$, where $A = \{a_1, \ldots, a_n\} \subseteq G$, and γ is a path with $\gamma^* \subseteq G\backslash A$:

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)\,dz}{f(z)} = \sum_{j=1}^{n} O(f, a_j)n(\gamma, a_j)$$

where $n(\gamma, w) = \dfrac{1}{2\pi i}\displaystyle\int_\gamma \frac{dz}{z-w}$.

If γ^* has a single bounded component B on which $n(\gamma, w) = 1$, then:

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)\,dz}{f(z)} = \sum_{a_j \in B} O(f, a_j)$$

10 Jordan lemma

- **Jordan lemma**: If $|f(z)| < \frac{M}{R^c}$ for $z = Re^{i\theta}$, where $c > 0$ and M are constants, then:

$$\lim_{R\to\infty}\int_S e^{iaz} f(z)\,dz = 0$$

where S is the semicircle $z = Re^{i\theta}$, for $0 \le \theta \le \pi$, $R > 0$ constant.

11 Logarithm function

- For $z = x + iy$, $e^z = e^x(\cos y + i \sin y)$.
- It is 2π-periodic in y, and thus any strip of height 2π will suffice. The principle branch is $-\pi < y \le \pi$.
- Logarithm function: $\log_{(0)} z = \log|z| + i \arg z$, $\log_{(1)} z = log\,|z| + i(\arg z + 2\pi)$
- Exponential function: $z^c = e^{c \log z}$
- The square root function is $f(z) = z^{\frac{1}{2}} = \sqrt{r}e^{\frac{i\theta}{2}}$. There are branch points at 0 and ∞. It is a two-valued function since each time a closed path goes around $z = 0$, the square root is multiplied by -1.