# MATH3964 Summaries (Part 3)

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## 1 Gamma function

- **Gamma function**:
$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$
This is analytic for $\operatorname{Re} z > 0$.
- For $\operatorname{Re} z > 1$:
$$\Gamma(z) = (z-1)\Gamma(z-1)$$
- $\Gamma(1) = 1$, $\Gamma(n+1) = n!$ for integer $n$.
- **Hankel's contour**: goes from $+\infty$ to $+\delta$ just above the real axis, goes around $C_\delta$ anti-clockwise, and then returns to $+\infty$ just below the real axis.
- If $C$ is Hankel's contour, for $\operatorname{Re} z > 0$:
$$\Gamma(z) = \frac{i}{2\sin \pi z} \int_C e^{-w}(-w)^{z-1}\, dw$$
There is a branch cut on $[0, \infty)$, and this formula gives analytic continuation for all $z$.
- The continued gamma function has poles of order 1 at $z = 0, -1, -2, \ldots$ (from the zeroes of $\pi z$) with residue $\dfrac{(-1)^n}{n!}$.
- $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$

## 2 Riemann zeta function

- **Riemann zeta function**: For $\operatorname{Re} s > 1$:
$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$$
- **Euler product**: For $\operatorname{Re} s > 1$:
$$\sum_{n=1}^\infty \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

- Lemmas:
  - Let $S = \{z : \operatorname{Re} z \geq a\}$ where $a > 1$. $\forall \epsilon > 0$, $\exists\, \delta, 0 < \delta < 1$, such that $\forall z \in S$:
  $$\left| \int_{\alpha}^{\beta} (e^t - 1)^{-1} t^{z-1}\, dt \right| < \epsilon$$
  if $0 \leq \alpha < \beta < \delta$.
  - Let $S = \{z : \operatorname{Re} z \leq A\}$ where $-\infty < A < \infty$. $\forall \epsilon > 0$, $\exists K > 1$ such that $\forall z \in S$:
  $$\left| \int_{\alpha}^{\beta} (e^t - 1)^{-1} t^{z-1}\, dt \right| < \epsilon$$
  if $k < \alpha < \beta \leq \infty$.
  - The integrals:
  $$\int_{\alpha}^{1} (e^t - 1)^{-1} t^{z-1}\, dt$$
  satisfies a Cauchy criterion as $\alpha \to 0$. The difference between any two will be arbitrarily small if $\alpha$ and $\beta$ are taken sufficiently close to 0.
  - In the integrals:
  $$\int_{1}^{\beta} (e^t - 1)^{-1} t^{z-1}\, dt$$
  take $1 < \beta \to \infty$, which implies the convergence of:
  $$\int_{1}^{\infty} (e^t - 1)^{-1} t^{z-1}\, dt$$
  - If $S = \{z : a \leq \operatorname{Re} z \leq A\}$ where $1 < a < A < \infty$, then:
  $$\int_{0}^{\infty} (e^t - 1)^{-1} t^{z-1}\, dt$$
  converges uniformly on $S$, and defines an analytic function.
  - If $S = \{z : \operatorname{Re} z \leq A\}$ where $-\infty < A < \infty$, then:
  $$\int_{1}^{\infty} (e^t - 1)^{-1} t^{z-1}\, dt$$
  converges uniformly on $S$.

- For $\operatorname{Re} z > 1$:
$$\Gamma(z)\zeta(z) = \int_{0}^{\infty} (e^t - 1)^{-1} t^{z-1}\, dt$$
This is analytic in $z$ for $\operatorname{Re} z > 1$.

- If $C$ is Hankel's contour ($0 < \delta < 2\pi$), $\forall s \in \mathbb{C}$:
$$I(s) = \int_{C} (e^w - 1)^{-1} (-w)^{s-1}\, dw$$
is an analytic function of $s$.

- Moreover, for $\operatorname{Re} s > 1$:
$$\zeta(s) = \frac{i\Gamma(1-s)}{2\pi} I(s)$$
Since $I(s)$ is analytic for all $s$ and $\Gamma(1-s)$ is a well-defined meromorphic function for all $s$, this gives the analytic continuation of $\zeta(s)$ to the whole complex plane.

- By examining the extended $\zeta$-function in $\operatorname{Re} s \leq 1$, the only poles will come from $\Gamma(1-s)$ since $I(s)$ is analytic. These occur at $s = 1, 2, 3, \ldots$, but $\zeta(s)$ is known to be analytic at $s = 2, 3, \ldots$. Hence, the only possible pole of $\zeta(s)$ is at $s = 1$. This also implies that $I(s) = 0$ for $s = 2, 3, \ldots$ to cancel out these poles.

- The residue of the pole of order 1 at $s = 1$ is 1.

- **Riemann Hypothesis:** $0 \leq \operatorname{Re} s \leq 1$ and $\zeta(s) = 0 \Rightarrow \operatorname{Re} s = \frac{1}{2}$

- $\zeta(s)$ is intimately connected to prime numbers:
  - Let $\pi(x)$ be the number of primes not exceeding $x$. By definition, $\pi(n) - \pi(n-1)$ is 1 if $n$ is prime, and 0 if $n$ is not prime.
  - **Prime number theorem:** $\pi(x) \sim \dfrac{x}{\log x}$
  - The Riemann Hypothesis is equivalen to $\pi(s) = \operatorname{ls} x + O(x^{1/2} \log^2 x)$, where $\operatorname{ls} x = \displaystyle\sum_{2 \leq n \leq x} \frac{1}{\log n} \geq \frac{n-1}{\log n}$
  - For $\operatorname{Re} s > 1$, $\log \zeta(s) = s \displaystyle\int_2^\infty \frac{\pi(x)}{x(x^s - 1)}\, dx$

## 3 Beta function

- $B(p,q) = \displaystyle\int_0^1 x^{p-1}(1-x)^{q-1}\, dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ for $p, q > 0$.

## 4 Weierstrass factorisation

- Given a sequence $\{u_n\}_{n=1}^\infty$ of complex numbers, $\displaystyle\sum_{n=1}^\infty |u_n|$ converges (or diverges) iff $\displaystyle\prod_{n=1}^\infty (1 + |u_n|)$ (absolutely) converges (or diverges).

- If an entire function $f(z)$ has no zeroes, then there is an entire function $\varphi(z)$ such that $f(z) = e^{\varphi(z)}$.

- If an entire function $f(z)$ has a finite number of zeroes at $z = 0, a_1, \ldots, a_n$ of order $m, m_1, \ldots, m_n$ respectively, then:
$$h(z) = \frac{f(z)}{z^m \left(1 - \frac{z}{a_1}\right)^{m_1} \cdots \left(1 - \frac{z}{a_n}\right)^{m_n}}$$

is an entire function without zeroes. Hence, $h(z) = e^{\varphi(z)}$ for some entire function $\varphi(z)$.

- Consider an infinite sequence $\{a_n\}_{n=1}^{\infty}$, and suppose that $0 < |a_n| \le |a_{n+1}|$ and $\lim_{n\to\infty} |a_n| = \infty$. Then for any real $R > 0$, there exists a sequence $k_1, k_2, \ldots$ of non-negative integers such that $\displaystyle\sum_{n=1}^{\infty} \left(\frac{R}{|a_n|}\right)^{k_n+1}$ is convergent.
  - For $n \ge 2$, put:

$$\begin{aligned} P_n(z) &= \frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{k_n}\left(\frac{z}{a_n}\right)^{k_n} \\ Q_n(z) &= \log\left(1 - \frac{z}{a_n}\right) + P_n(z) \\ E_n(z) &= \left(1 - \frac{z}{a_n}\right) e^{P_n(z)} = e^{Q_n(z)} \end{aligned}$$

  - For any fixed real $R > 0$, choose a positive integer $N$ such that $|a_n| > 2R$ for $n \ge N$. Then, $\displaystyle\prod_{n=N}^{\infty} E_n(z) = e^{\sum_{n=N}^{\infty} Q_n(z)}$ is holomorphic on $|z| \le R$ and converges absolutely.
  - The only zeroes of $g(z) = \left(1 - \dfrac{z}{a_n}\right) \displaystyle\prod_{n=2}^{\infty} e^{P_n(z)}$ are $a_n$ $(n = 1, 2, \ldots)$ counted with multiplicity.
  - **Weierstrass factorisation:** There exists an entire function $h(z)$ such that:

$$f(z) = z^m \cdot \left(1 - \frac{z}{a_n}\right) \left(\sum_{n=2}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{P_n(z)}\right) e^{h(z)}$$