MATH3964 Summaries (Part 4)

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1 Theta functions

- A function is **double periodic** if there exist two linearly independent complex numbers ω_1 and ω_2 such that $f(z + n_1\omega_1 + n_2\omega_2) = f(z)$ for all $n_1, n_2 \in \mathbb{Z}$.
- By Liouville's theorem, if a double periodic function is everywhere analytic, then f is constant. If you allow poles, you will get elliptic functions such as Weierstrass $\wp$ functions, and if you modify the periodicity, you get theta functions.
- **Theta functions:** Fix $\tau \notin \mathbb{R}$. We want the function ϑ to satisfy $\vartheta(z+1) = \vartheta(z)$ and $\vartheta(z+\tau) = F(z)\vartheta(z)$ for some F.
- One non-constant choice of F is $F(z) = ce^{-2\pi i z}$, $c \neq 0$.
- Expanding $\vartheta(z)$ as a Fourier series, $\vartheta(z) = \sum_{n=-\infty}^{\infty} A_n e^{2\pi i n z}$. This results in $A_n = c^{-n} e^{\pi i n(n-1)\tau}$.
- By taking $q = e^{\pi i \tau}$, $c = q^{-1}$, $A_n = q^{n^2}$ and $\text{Im } \tau \geq 0$:

$$\begin{aligned}
\vartheta(z,\tau) &= \sum_{n=-\infty}^{\infty} q^{n^2} e^{2\pi i n z} \\
\vartheta(z+1,\tau) &= \vartheta(z,\tau) \\
\vartheta(z+\tau,\tau) &= q^{-1} e^{-2\pi i z} \vartheta(z,\tau)
\end{aligned}$$

- $\vartheta(z)$ converges absolutely uniformly in compact subsets of the complex plane.
- Transformation formulas: Let $\vartheta_{a,b}(z,\tau) = \sum q^{\left(n+\frac{a}{2}\right)^2} e^{2\pi i \left(n+\frac{a}{2}\right)\left(z+\frac{b}{2}\right)}$.
- Let $\Psi(y,\tau) = e^{\pi i c(c\tau+d)y^2} \vartheta\big((c\tau+d)y, \tau\big)$:
  - $\Psi(y+1,\tau) = \Psi(y,\tau)$
  - $\Psi\left(y + \frac{a\tau+b}{c\tau+d}, \tau\right) = e^{-\pi i \frac{a\tau+b}{c\tau+d} - 2\pi i y} \Psi(y,\tau)$
  - $\Psi(y,\tau) = \varphi(\tau)\vartheta\left(y, \frac{a\tau+b}{c\tau+d}\right)$ for some $\varphi(\tau)$ independent of z

- The **zeroes** of $\vartheta_{11}(z,\tau)$ in z are all of the form $m_1 + m_2\tau$ for $m_1, m_2 \in \mathbb{Z}$ and they are of order 1. There is only one zero in the parallelogram with edges formed by vectors 1 and τ.

- $\vartheta_{00}(z,\tau)$ has zeroes at $z = n + \frac{1}{2} + \left(m + \frac{1}{2}\right)\tau$.

- Use the zeroes of ϑ_{00} to construct $f(z) = \prod_{m=1}^{\infty}\left(1 + q^{2m-1}e^{2\pi i z}\right)\left(1 + q^{2m-1}e^{-2\pi i z}\right)$, which converges absolutely uniformly.

- f and ϑ_{00} have the same quasi-periodic properties, so $\frac{\vartheta_{00}(z)}{f(z)} = T(q)$ has period 1 and τ, and is independent of z.

- Then, $T(q) = \prod_{m=1}^{\infty}\left(1 - q^{2m}\right)$, and hence:

$$\vartheta_{00}(z,\tau) = \prod_{m=1}^{\infty}\left(1 - q^{2m}\right)\left(1 + q^{2m-1}e^{2\pi i z}\right)\left(1 + q^{2m-1}e^{-2\pi i z}\right)$$

- **Jacobi's formula for null theta values:**

$$-\pi\vartheta_{00}(0,\tau)\vartheta_{01}(0,\tau)\vartheta_{10}(0,\tau) = \vartheta_{11}'(0,\tau)$$

2 Elliptic functions

- Choose two linearly independent complex numbers ω_1 and ω_2. $L = \{n_1\omega_1 + n_2\omega_2 : n_1, n_2 \in \mathbb{Z}\}$ is the **lattice** in $\mathbb{C}$ spanned by ω_1 and ω_2.

- If P is the parallelogram with vertices $0, \omega_1, \omega_2, \omega_1 + \omega_2$, then $\forall\, z \in \mathbb{C}, \exists\, \omega \in L$ such that $z - \omega \in P$.

- For a lattice L, $\sum_{0 \neq \omega \in L} \frac{1}{|\omega|^a} < \infty$ for $a > 2$. Also, $\sum_{0 \neq \omega \in L}\left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right)$ converges absolutely uniformly on compact subsets of $\mathbb{C}$.

- A meromorphic function f is **elliptic** with respect to a given lattice L if $\forall\, \omega \in L$, $f(z+\omega) = f(z)$. The value of the function f is completely determined by its value on P. If the function had no poles, it would be constant.

- Poles in an elliptic function are isolated, and the sum of residues of an elliptic function is zero.

- Suppose $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are the poles and zeroes of $f(z)$ respectively, counted according to multiciplicity and all lying within a period parallelogram. Then $a_1 + \ldots + a_n \equiv b_1 + \ldots + b_n \mod L$, where L is a lattice.

- The **Weierstrass $\wp$ function** is:

$$\wp(z) = \frac{1}{z^2} + \sum_{0 \neq \omega \in L} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

  $\wp$ is elliptic with period lattice L, and converges absolutely uniformly on compact subsets of $\mathbb{C}$.

- Applying $\left(\sum_{n=0}^{\infty} x^n \right)^2 = \sum_{m=0}^{\infty} (m+1)x^m$ and the fact that $\sum_{0 \neq \omega \in L} \frac{1}{w^n} = 0$ if n is odd, we get:

$$\wp(z) = \frac{1}{z^2} + \sum_{k=2}^{\infty} (2k-1) G_{2k} z^{2k-2}$$

  where $G_{2k} = \sum_{0 \neq \omega \in L} w^{-2k}$.

- By comparing coefficients, we get:

$$\wp'(z)^2 = 4\wp(z)^3 - g_2 \wp(z) - g_3$$

  where $g_2 = 60G_4$ and $g_3 = 140G_6$.

- $\wp(z)$ is a solution of the differential equation $\left(\frac{dw}{dz}\right)^2 = 4w^3 - g_2 w - g_3$, where $g_2^3 - 27g_3^2 \neq 0$. Rearranging, this implies:

$$\wp^{-1}(x) = \int \frac{dx}{\sqrt{4x^3 - g_2 x - g_3}}$$

- The algebraic equation $y^2 = 4x^3 - g_2 x - g^3$, where $g_2^3 - 27g_3^2 \neq 0$, is an **elliptic curve**, which is the image of the map $z \mapsto \big(\wp(z), \wp'(z)\big)$ according to the differential equation $\big(\wp'(z)\big)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$.

- We write $z_1 \equiv z_2$ if $z_1 - z_2 \in L$. $[z]$ is the equivalence class of z, and $\mathbb{C}/L$ is the set of all equivalence classes.

- The map $\overline{p} : [z] \mapsto \big(\wp(z), \wp'(z)\big)$ is a bijection.

- $\mathbb{C}/L$ is an abelian group under the addition of complex numbers, that is, $[u_1] + [u_2]$ is defined to be $[u_1 + u_2]$. There is an addition $\dot{+}$ on the elliptic curve such that $\overline{p}$ is a group homomorphism, that is, $\overline{p}\big([u_1] + [u_2]\big) = \overline{p}\big([u_1]\big) \dot{+} \overline{p}\big([u_2]\big)$.

- The addition law $\dot{+}$ is given by rational functions. If $P_j(x_j, y_j)$, for $j = 1, 2, 3$, and $P_3 = P_1 \dot{+} P_2$, then:

$$\begin{aligned} x_3 &= -x_1 - x_2 + \frac{1}{4}\left(\frac{y_1 - y_2}{x_1 - x_2}\right)^2 \\ y_3 &= \frac{x_1 y_1 - x_2 y_2}{x_1 - x_2} - \frac{1}{4}\left(\frac{y_1 - y_2}{x_1 - x_2}\right)^3 \end{aligned}$$

These results come from:

$$\begin{aligned}\wp(u_1+u_2) &= -\wp(u_1)-\wp(u_2)+\frac{1}{4}\left(\frac{\wp'(u_1)-\wp'(u_2)}{\wp(u_1)-\wp(u_2)}\right)^2 \\ \wp'(u_1+u_2) &= -\frac{\wp'(u_1)-\wp'(u_2)}{\wp(u_1)-\wp(u_2)}\wp(u_1+u_2)+\frac{\wp(u_1)\wp'(u_2)-\wp'(u_1)\wp(u_2)}{\wp(u_1)-\wp(u_2)}\end{aligned}$$

3 Jacobi elliptic functions

- **Jacobi elliptic sine function:** $\mathrm{sn}(z,k)=\mathrm{sn}(z)$ is defined by the following:

$$\mathrm{sn}^{-1}\,w=\int_0^w \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}$$

or:

$$z=\int_0^{\mathrm{sn}(z)} \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}$$

where $0<k<1$ is the modulus.

- Modulus:

$$K=K(k)=\int_0^1 \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}$$

- Complementary modulus:

$$K'=K'(k)=\int_0^1 \frac{dt}{\sqrt{1-t^2}\sqrt{1-(1-k)^2t^2}}$$

- **Jacobi elliptic cosine function:** $\mathrm{cn}(z)$ and $\mathrm{dn}(z)$ are defined by the following:

$$\mathrm{cn}(z)=\sqrt{1-\mathrm{sn}^2(z)} \qquad \mathrm{dn}(z)=\sqrt{1-k^2\,\mathrm{sn}^2(z)}$$

- These functions are generalisations of the trigonometric functions:

$$\begin{aligned}\mathrm{sn}(z,0) &= \sin z \\ \mathrm{cn}(z,0) &= \cos z \\ \mathrm{dn}(z,0) &= 1\end{aligned}$$

- Take branch cuts for $\mathrm{sn}\,z$ from $-\frac{1}{k}$ to -1, and from 1 to $\frac{1}{k}$.
- $\mathrm{sn}\,z$ is an odd function.
- $\mathrm{sn}\,z$, $\mathrm{cn}\,z$ and $\mathrm{dn}\,z$ are periodic in $K(k)$ and $K'(k)$:

$$\begin{aligned}\mathrm{sn}(z+2mK+2niK') &= (-1)^m\,\mathrm{sn}\,z \\ \mathrm{cn}(z+2mK+2niK') &= (-1)^{m+n}\,\mathrm{cn}\,z \\ \mathrm{dn}(z+2mK+2niK') &= (-1)^n\,\mathrm{dn}\,z\end{aligned}$$

- $\operatorname{sn} z$ satisfies the following differential equation:

$$\left(\frac{dw}{dz}\right)^2 = (1-w^2)(1-k^2w^2)$$

where $w = \operatorname{sn} z$.

- Special values:

$$\begin{aligned}
\operatorname{sn} 0 &= 0 \\
\operatorname{cn} 0 &= 1 \\
\operatorname{dn} 0 &= 1 \\
\operatorname{sn}\big(K(k)\big) &= 1
\end{aligned}$$

- Identities:

$$\begin{aligned}
\operatorname{sn}^2 z + \operatorname{cn}^2 z &= 1 \\
k^2 \operatorname{sn}^2 z + \operatorname{dn}^2 z &= 1 \\
k^2 \operatorname{cn}^2 z + k'^2 &= \operatorname{dn}^2 z \\
\operatorname{cn}^2 z + k'^2 \operatorname{sn}^2 z &= \operatorname{dn}^2 z
\end{aligned}$$

- Differentiating:

$$\begin{aligned}
\frac{d}{dz}\operatorname{sn} z &= \operatorname{cn} z \operatorname{dn} z \\
\frac{d}{dz}\operatorname{cn} z &= -\operatorname{sn} z \operatorname{dn} z \\
\frac{d}{dz}\operatorname{dn} z &= -k^2 \operatorname{sn} z \operatorname{cn} z
\end{aligned}$$

- Taylor series for Jacobi elliptic functions:

$$\begin{aligned}
\operatorname{sn} z &= z - \frac{1+k^2}{6}z^3 + \left(1+14k^2+k^4\right)\frac{z^5}{5!} - \cdots \\
\operatorname{cn} z &= 1 - \frac{z^2}{2} + \left(1+4k^2\right)\frac{z^4}{4!} - \cdots \\
\operatorname{dn} z &= 1 - \frac{k^2z^2}{2} + k^2\left(4+k^2\right)\frac{z^4}{4!} - \cdots
\end{aligned}$$

- Because $\operatorname{sn}(z+iK') = \dfrac{1}{k \operatorname{sn} z}$ and $z=0$ is a zero of order 1 of $\operatorname{sn} z$, iK' is a pole of order 1 of $\operatorname{sn} z$ with residue $\frac{1}{k}$, and also at $(2n+1)iK'$, $n \in \mathbb{Z}$.

- If $P(z)$ is a polynomial with degree 3 or 4, it is possible to change integrals of the form $\displaystyle\int_{\mathscr{C}} \frac{dz}{\sqrt{P(z)}}$ into an elliptic integral by a change of variable.

- Addition formulas:

$$\begin{aligned}\operatorname{sn}(x+y) &= \frac{\operatorname{sn} x \operatorname{cn} y \operatorname{dn} y + \operatorname{sn} y \operatorname{cn} x \operatorname{dn} x}{1 - k^2 \operatorname{sn}^2 x \operatorname{sn}^2 y} \\ \operatorname{cn}(x+y) &= \frac{\operatorname{cn} x \operatorname{cn} y - \operatorname{sn} x \operatorname{sn} y \operatorname{dn} x \operatorname{dn} y}{1 - k^2 \operatorname{sn}^2 x \operatorname{sn}^2 y} \\ \operatorname{dn}(x+y) &= \frac{\operatorname{dn} x \operatorname{dn} y - \operatorname{sn} x \operatorname{sn} y \operatorname{cn} x \operatorname{cn} y}{1 - k^2 \operatorname{sn}^2 x \operatorname{sn}^2 y}\end{aligned}$$

4 Topology

- Let $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^3 + x_2^3 + x_3^3 = 1\}$. The north pole is at $(0, 0, 1)$ and the complex plane is identified with $x_3 = 0$.
- For a point $Q = (x_1, x_2, x_3) \in S\backslash\{N\}$, extend the line from N to Q until it meets the complex plane at $P(x, y, 0)$.
- This gives a map $\pi : S\backslash\{N\} \mapsto \mathbb{C}$:

$$\begin{aligned}\pi(x_1, x_2, x_3) &= \frac{x_1}{1 - x_3} + i\frac{x_2}{1 - x_3} \\ \pi^{-1}(x, y) &= \left(\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, 1 - \frac{2}{x^2 + y^2 + 1}\right)\end{aligned}$$

- Let $U'_{\infty,R} = \{z \in \mathbb{C} : |z| > R\}$. Then, $\pi^{-1}\left(U'_{\infty,R}\right) = \{(x_1, x_2, x_3) \in S\backslash\{N\} : x_3 > \frac{R-1}{R+1}\}$.
- We extend the complex plane to $\mathbb{C} \cup \{\infty\}$, and set $\pi(N) = \infty$ so that we have a bijection $\pi : S \mapsto \mathbb{C} \cup \{\infty\}$.
- The sphere S is topologically equivalent to $\mathbb{C} \cup \infty$, but S is closed and bounded, and thus $\mathbb{C} \cup \infty$ is also closed and bounded.
- Let Σ be a topological space (a set of open sets). Suppose that Σ is covered by a family of open sets U_i, such that for each i, there exists a homeomorphism (bijective continuous map) $\Phi_i : U_i \mapsto W_i$, where W_i is an open subset of $\mathbb{C}$. We can such a set of pairs $A = \big\{(U_i, \Phi_i)\big\}$ an **atlas** of Σ.
- (U_i, Φ_i) is a **chart** at s if $s \in U_i$, and $z_i = \Phi_i(s)$ is a **local coordinate**.